Degree Project in Engineering Physics

Second Cycle 30 Credits

On the emergent relative distance between quantum systems

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Stockholm, Sverige 2022

Abstract

In an emergent spacetime framework where relative distances between quantum systems are determined by the mutual information between the systems, an entangled pair must be shown to have a non-zero distance even though its mutual information is maximal by virtue of being maximally entangled. This report shows that in fact an entangled pair is only maximally entangled in some degrees of freedom such as spin, and when one introduces other quantum degrees of freedom the mutual information is no longer maximal and so a non-zero distance can be recovered. In light of a conjectured relationship between Einstein–Rosen Bridges (ER) and entangled Einstein–Podolsky–Rosen (EPR) pairs called ER = EPR, what appeared as a wormhole forming between the two quantum systems was an artefact of our ignorance of all quantum information associated with the systems.

Sammanfattning

I ett ramverk för emergent rumstid där relativa avstånd mellan kvantsystem bestäms av ömsesidig information mellan systemen måste ett sammanflätat par påvisas ha ett nollskilt avstånd även fast dess ömsesidiga information är maximal eftersom de är maximalt sammanflätade. Detta examensarbete visar att ett sammanflätat par är endast maximalt sammanflätade i vissa frihetsgrader som spinn, och när man introducerar andra frihetsgrader så är ömsesidiga informationen inte längre maximal och således kan man få ett nollskilt avstånd. Med hänsyn till ett förmodat förhållande mellan Einstein–Rosen bryggor (ER) och sammanflätade Einstein–Podolsky–Rosen par som benämns ER = EPR, det som framstår vara ett maskhål mellan två kvantsystem var således ett artefakt av vår ignorans för all kvantinformation associerat med systemen.

Acknowledgements

I would like to thank my supervisor at Nordita, Guilherme Franzmann, for the engaging discussions and fascinating insights in to the world of spacetime emergence and the foundations of quantum mechanics. This topic has been somewhat of a dream of mine for many years and I am very grateful to have found someone to share such a scientific passion with.

I would also like to extend my gratitude to my wife, family and friends who patiently sat through my attempts at understanding and conveying the contents of my thesis, but also for supporting me outside of my work.

I would finally like to thank my supervisor at KTH, Prof. Tommy Ohlsson, for proof-reading and helping me clarify my work.

Contents

1 Introduction

Reconciling Gravity with Quantum Mechanics (QM) is yet an open problem in physics. Ultimately, we would like to have a theory which accurately produces the results of either theory at the appropriate limits. Quantum theories have developed over the last century to accurately describe and predict phenomena, in particular quantum field theory (QFT) which underpins the Standard Model (SM) of particle physics. One such QFT is quantum electrodynamics, where a classical theory of electromagnetism was quantised by use of quantum fields with infinite degrees of freedom, allowing for predictions to test the SM, in particular of the fine structure constant α, with relative uncertainties of the order 10^{-12} [1]. By identifying more symmetries in nature the Strong and Weak forces were also quantised leading to the development of quantum chromodynamics and quantum flavour dynamics, respectively, the last subsumed alongside quantum electrodynamics under the electroweak unification.

In all the above cases, since we usually integrate over the entire momentum space in our theories, the integrals for the perturbative corrections lead to divergencies. Naturally one may then consider that there exist field theories that are only valid at certain scales called effective field theories (EFT) which in general are non-renormalisable. The most notable example is Fermi's beta decay interaction which was later replaced by the Weak interaction. To tame these divergencies, one may introduce cutoffs to the integrals, called regularisation, where new field theories are required beyond the cutoffs. The renormalisation group was then introduced to probe this question and a plethora of renormalisation techniques to tame the divergencies of field theories have been developed.

Three out of four fundamental forces have been successfully quantised *and* renormalised, so what about gravity? The best description of gravity so far, General Relativity (GR), is a non-renormalisable theory[2, 3] since the coupling has mass dimension $[G_N] = 2 - d$, meaning a negative mass dimension for $d > 2$, and it is likely that GR is an EFT of a more fundamental theory [4, 5].

The attempt to reconcile GR with QM is called Quantum Gravity (QG), and there are numerous approaches, the primary two being String Theory and Loop Quantum Gravity. A conjecture formed within the field of string theory called AdS/CFT correspondence [6] provided a promising avenue for the development of a theory of QG. More specifically, the maximally symmetric Lorentz manifold with negative curvature, Anti de Sitter space, in 5 dimensions, AdS_5, has the isometry group $SO(2,4)$ which is the conformal group for flat Minkowski space $\mathbb{R}^{1,3}$ [7]. In other words, for a bulk region described by AdS_5, the the 4-dimensional boundary is described by a Conformal Field Theory (CFT) living in flat Minkowski spacetime. The two descriptions are dual and completely specified by one another. This correspondence has also been called Gauge/Gravity duality and has been explored more generally since its first conception.

Another approach to reconciling gravity with QM is actually subsuming gravity into QM by proposing that spacetime is an emergent phenomenon of quantum systems. A conjectured relationship between Einstein–Rosen (ER) bridges and Einstein–Podolsky–Rosen (EPR) pairs called ER=EPR [8] suggests the following: not only is there an equivalence between a wormhole connecting two spacetime regions and a pair of maximally entangled black holes but between any entangled system. The overall approach follows a long thread of work that started with the findings by Bekenstein [9], exploring the thermodynamic properties of black holes and relating the entropy of a black hole to its surface area. This led to the suggestion of the holographic principle by 't Hooft [10], which states that the information in a bulk region of space can be encoded on the boundary of this region. Maldacena [6] used this to develop the conjectured AdS/CFT correspondence, a holographic relationship between anti-de Sitter space in the bulk and a conformal field theory on the boundary. Ryu and Takyanagi [11] then showed that in the case of an anti-de Sitter spacetime, there is a direct relationship between the entanglement entropy associated with bulk regions separated by a boundary surface where a conformal field theory is defined and the area of this boundary. This was then furthered by van Raamsdonk [12], suggesting that one could relate the boundary surface between bulk regions and the distance between them in an AdS/CFT setting.

More recently, Cao et al. [13, 14] combine much of this approach into a new research program with the goal of starting from a purely quantum mechanical framework and its entanglement structure and deriving classical spacetime geometry satisfying Einstein's equations. In short, the program suggests that there is a mapping between the mutual information of quantum subsystems and the classical geometry connecting them, giving rise to an emergent spacetime purely defined in terms of the quantum information contained in the system. In this approach, we can maintain Einstein's own interpretation of gravitational fields as intrinsically connected to spacetime geometry itself, where now the spacetime geometry would be derived from the entanglement structure of quantum systems.

One problem with this approach is that an entangled EPR pair, for instance, will always be seen as having zero distance between its subcomponents, since it is maximally entangled, despite the fact that its subcomponents can be spacelike separated forming what so far has been called a wormhole between the pair. No wormhole has to date been observed so this interpretation must be incomplete. In this report we show that once one includes other quantum degrees of freedom for the systems and compare how they decohere with respect to an environment, the mutual information is no longer maximal and so a metric based on mutual information between an EPR pair will in fact be non-maximal, corresponding to a non-zero distance between them. Therefore the appearance of a wormhole between subsystems is only apparent due to the ignorance of all quantum degrees of freedom.

We begin with a review of ER = EPR in Sec. 2.1, presenting the thread of research from the original papers by Einstein, Rosen and Podolsky up to the paper published by Susskind and Maldacena on the conjecture influenced by AdS/CFT. Then in Sec. 2.2 spacetime emergence is reviewed, in particular an overview of the framework proposed by Cao et al. [13] within which the research of this report is focused. After that the mutual information of entangled pairs of increasing complexity are treated. In Sec. 3.1 a simple EPR pair is presented and its mutual information computed. The result is then generalised for the GHZ state in Sec. 3.2. Next the environment is introduced for a simple EPR pair, first unentangled in Sec. 3.3 and then entangled in Sec. 3.4 including the involvement of monogamy of entanglement. Decoherence is then introduced in Sec. 3.5 w.r.t how mutual information changes as an entangled pair interacts with its environment. A more realistic case of EPR pairs is then considered in Sec. 3.6, introducing momentum degrees of freedom. This is all brought together in Sec. 3.8 where the effects on the environment itself is regarded i.e the environment's entanglement structure changing as a result of the entangled pairs decohering. Some additional points of interests are then discussed in Sec. 4. In Sec. 4.1, the preferred basis problem is introduced, since any entanglement structure will depend on the choice of global basis for the Hilbert space. In Sec. 4.2, an outline of the problem and resolution to finite representations of momentum and position is presented and how the framework could be possibly extended to infinite-dimensional cases. Sec. 4.3 introduces an important word of warning by Verlinde in regards to ER=EPR. A summary of the work is finally presented in Sec. 5, concluding that in an emergent spacetime framework based on mutual information, an EPR pair would not necessarily yield a zero relative distance.

The report relies on the use of Bra-Ket notation for states and due to the size of some states, a notation for contracting indices is introduced. In particular, for a state given by some indices $i_1, ..., i_n$ such as $|i_1, ..., i_n\rangle$, the contracted capital index I is used to refer to the set of indices $i_1, ...i_n$. Therefore, a sum over indices $i_1, ..., i_n$ can instead be written as a sum over I. This is explained in detail in Appendix A.2. Tensor products are also used, where the symbol $\otimes$ refers to the Kroenecker product, meaning that for some matrices A and B being $m \times n$ and $p \times q$, respectively, $A \otimes B$ is $mp \times nq$.

2 Basics

This section aims to review ER = EPR by reviewing relevant parts of the original papers published by Einstein, Rosen and Podolsky. Susskind and Maldacena's argument for the conjectured relationship between the two is then presented from an AdS/CFT point of view. The conjectured emergence of spacetime is then presented, along with an introduction on the approach Cao et al. make to recover spacetime satisfying Einstein's equation from quantum mechanics.

2.1 ER = EPR

The first example of entanglement was considered in the seminal paper by Einstein, Podolsky and Rosen (EPR) in 1935 [15]. They attempted to argue that Quantum Mechanics as a theory of reality was incomplete as it led to apparent contradictions according to their definitions of reality. In particular, they demanded the following criterion of a theory of reality, the first being completeness [15]

> every element of the physical reality must have a counterpart in the physical theory

and the second, later called the criterion of local realism [15]

> If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

With this in mind, they consider two initially interacting systems, I and II, with a corresponding wave function Ψ that stops interacting after some time T. They then consider two non-commuting observables A and B representing momentum and position, respectively, and consider the measurement on system I w.r.t. to each observable. In the Copenhagen interpretation employed in the paper, the act of measuring system I collapses the state Ψ, allowing us to assign a particular state to system II, from which we can identify the states being eigenfunctions of operators P and Q corresponding to the momentum and position of the second state. By performing successive measurements of two non-commuting observables on system I, we can assign values of momentum and position to the second system. Since the second system has never been measured, by the criterion of locality, both quantities must correspond to the same physical reality i.e. the successive measurement of two non-commuting observables of system II has been performed without perturbing the system, hence a contradiction has been reached. The authors admit [15]

> [one] could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as simultaneous elements of reality *only when they can be simultaneously measured or predicted.* On this point of view, since either one or the other, but not simultaneously, of the quantities P and Q can be predicted, they are not simultaneously real. This makes the reality of P and Q depend upon the process of measurement carried out on the first system which does not disturb the second system in any way. No reasonable definition of reality could be

expected to permit this.

It was Schrödinger later in the same year who coined the term "entanglement" to describe this correlated phenomenon [16]. Although this behaviour appears non-local w.r.t. information travelling superluminally, the No Communication Theorem prohibits the transmission of superluminal signals or in fact, the communication of any information at all by means of measuring entangled systems [17].

Only a month later during the same year, Einstein and Rosen (ER) wrote a paper initially intending to conceive of a universe described only by GR and EM, with space represented as two identical sheets connected by particles forming what they called "bridges"[18]. Starting from the spherically symmetric Schwarzschild metric representing a body of mass m

$$ds^2 = -\frac{1}{1-2\frac{m}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(1 - \frac{2m}{r}\right)dt^2\,, \tag{2.1.1}$$

written in natural units[1] they performed a parametrisation w.r.t. the radial coordinate r and mass m as $u^2 = r - 2m$ to remove the singularity at $r = 2m$, rewriting the metric as

$$ds^2 = -4(u^2+2m)du^2 - (u^2+2m)^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{u^2}{u^2+2m}dt^2\,. \tag{2.1.2}$$

By considering the cases $u > 0$ and $u < 0$ one gets two infinite "sheets" and a hyperplane joining the sheets at $u = 0$ and it is this joining hyperplane called the "bridge" that was later coined as a wormhole. If the two sheets of spacetime, representing asymptotically flat regions of spacetime, were separated by a large (or even infinite) distance, such a bridge would provide a path shorter than a lightlike path between two spacetime points as can be seen in Fig. 1, allowing for superluminal signals. Such a bridge would then be non-local, but Topological Censorship prohibits the traversability of wormholes in asymptotically flat spacetime [19].

Figure 1: Spacetime coordinates A and B on two asymptotically flat sheets connected by a lightlike path with a bridge between the two sheets.

It is only much later that Susskind and Maldacena proposed a conjectured correspondence between the two ideas, which they called ER = EPR [8]. In the AdS/CFT

[1] The Schwarzschild radius, the radius an object of mass m would have if it were a black hole, is given by $r_s = \frac{2Gm}{c^2}$. If one chooses natural units so that $c = G = 1$, the Schwarzschild radius is simply $r_s = 2m$.

framework, an eternal AdS-Schwarzschild black hole can be represented as a Thermofield Double (TFD) state

$$|TFD\rangle = \sum_n e^{-\frac{\beta E_n}{2}} |n, n\rangle \ , \tag{2.1.3}$$

where β is the inverse temperature of the black hole and the states are tensor products of eigenstates of two disconnected CFT's, often referred to by Left (L) and Right (R). An interpretation of this can be found by considering Wick rotating the state as two maximally entangled black holes in disconnected spaces with a common time. Being a pure state with an entropy of 0, computing the reduced entropy and noting that $S_L = S_R$ we find that the mutual information between them is nonclassical i.e. $2S_{BH}$, where S_{BH} is the Bekenstein–Hawking entropy of black holes given by $S_{BH} = \frac{A}{4G_N}$.

At the same time, we can in GR consider two black holes on separate sheets that can be non-locally connected by an ER bridge. Susskind and Maldacena then argue for the similarities between entangled black holes and black holes connected by an ER bridge, forming the conjecture that entanglement and ER bridges are two sides of the same coin, hence calling it ER = EPR. This can be seen by thinking of a very large set of EPR pairs given to Alice and Bob who compress them to a pair of black holes and comparing this to a case where a large number of pairs of black holes with ER bridges are given to Alice and Bob who collapse them into a large black hole pair with a single bridge.

2.2 Spacetime emergence

In an essay by Van Raamsdonk [12], a similar consideration is made as Susskind and Maldacena from the Holographic principles of AdS/CFT. An entangled pair of non-interacting CFT's written as a TFD state can be interpreted as classically disconnected w.r.t. spacetime. By then saying that this state corresponds exactly to an AdS eternal black hole connecting two asymptotic spacetime regions, one can infer that there is a relationship between disconnected CFT's and connected spacetimes, which he then rephrases as [12]

> classical connectivity arises by entangling the degrees of freedom in the two components.

As a simple example he considers a CFT on an S^d sphere with a corresponding asymptotically AdS spacetime in the bulk, partitioned into two hemispheres called A and B. With this partitioning, we partition the Hilbert space as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, allowing us to compute the mutual information between subsystems belonging to the two hemispheres. As shown by Ryu and Takayanagi [11], the entanglement entropy between two subsystems of a CFT in $d+1$ is related to the minimal surface in AdS_{d+2}, in other words, as the degrees of freedom in A and B disentangle in the boundary, the minimal surface connecting the hemispheres in the bulk decreases. This follows similar suggestions of an "area law", that entropy grows in proportion to the area and not the volume of the system [20, 21]. Introducing operators $\mathcal{O}_C$ and $\mathcal{O}_D$ for some boundary points $C \subset A$ and $D \subset B$, we can bound the correlations between C and D with the mutual information as [22]

$$I(C : D) \geq \frac{(\langle \mathcal{O}_C \mathcal{O}_D \rangle - \langle \mathcal{O}_C \rangle \langle \mathcal{O}_D \rangle)^2}{2||\mathcal{O}_C||^2||\mathcal{O}_D||^2} \ . \tag{2.2.1}$$

We can in AdS/CFT express the correlation length in terms of the 2-point correlator for some coefficient m with mass dimension 1 and shortest proper length L in the bulk between the boundary points C and D

$$\langle \mathcal{O}_C \mathcal{O}_D \rangle \sim e^{-mL}\,, \tag{2.2.2}$$

and knowing that the one-point correlators of a CFT vanish [23], we see that as $I(C : D) \to 0$ then the proper distance $L \to \infty$. In short, as the the degrees of freedom in A and B disentangle, the mutual information corresponding to the minimal surface between A and B decreases and the minimal proper length between boundary points in A and B increases, which he refers to as the spacetime regions of A and B "pinching off" from one another. The suggestion laid forth is that classically connected spacetimes could be a result of entanglement structures.

Cao, Carroll and Michalakis then formulated a tentative approach to recovering classical spacetime from entanglement structures [13]. By arguing from the holographic principle that any finite region of spacetime has finite dimensional degrees of freedom since $\dim \mathcal{H} \sim e^S$ where the entropy is given by the holographic bound, the general strategy goes as follows:

- decompose Hilbert space into factors $\mathcal{H} = \bigotimes_i \mathcal{H}_i$.
- consider only states that are "Redundancy Constrained" (RC) i.e. satisfying an area law $S = \eta A + ...$. If the states satisfy the area law, one can approximate the entropy of some larger region (a collection of subregions A_p in Hilbert space) $\mathbf{B}$ using a cut function based on mutual information

$$S(B) = \frac{1}{2} \sum_{p\in \mathbf{B}, q \in \bar{\mathbf{B}}} I(A_p : A_q)\,. \tag{2.2.3}$$

- construct a weighted graph G with vertices represented by the subregions and edges represented by mutual information $I(A_p : A_q)$ between subregions.
- reconstruct a metric graph $\tilde{G}$ with smooth, flat geometries from some mapping $G \mapsto \tilde{G}$.
- relate local curvature to local change in entropy by means of perturbations $|\psi_0\rangle \mapsto |\psi_0\rangle + |\delta\psi\rangle$.
- relate the change in entropy to the change of entropy of an effective field theory and recover the linearised Einstein field equations by means of the entanglement first law $\delta S = \delta \langle K \rangle$ relating the change in entropy to the modular Hamiltonian.

The metric graph $\tilde{G}$ is assumed to be a re-weighting of the edges of the graph by

$$w(A,B) = \begin{cases} l_{RC}\Phi(I(A_p : A_q)/I_0), & p \neq q \\ 0 & p = q\,, \end{cases} \tag{2.2.4}$$

for some function of mutual information $\Phi(I(A_p : A_q)/I_0)$ where $I_0 = \max\{I(A_p : A_q)\}$ and l_{RC} is the scale of RC states. The function Φ is defined to be a monotonically decreasing function such that $\lim_{x\to 1} \Phi(x) \to 0$, the mutual information between the vertex and itself, and $\lim_{x \to 0} \Phi(x) \to \infty$, minimal mutual information between vertices.

A suitable candidate would be $\Phi(x) = -\log(x)$, bearing in mind that it's true form may be more complicated to accommodate arbitrary spatial geometries.

The vertices p and q are connected by a large set of vertices with many possible paths, such as $P = \{p_0 = p, p_1, ..., p_{k-1}, p_k = q\}$, so the minimal path P giving the distance function $d(A, B)$ is chosen to be the path which minimises the sum of weights

$$d(A, B) = \min_P \left\{ \sum_{n=0}^{k-1} w(p_n, p_{n+1}) \right\} , \tag{2.2.5}$$

which by construction satisfies the properties of a metric, since mutual information is symmetric and positive and the minimisation of weights along a path satisfies the triangle inequality. In this construction, if one naïvely took a simple EPR pair and computed the mutual information between them, which would be maximal, one would find the shortest path between them to be zero, which we know to not be true in the lab because we can create entangled particles and separate them spatially. Another interpretation could be in the vein of ER = EPR that our metric giving $d(A, B) = 0$ implies we have formed a wormhole between A and B, also implying that the true metric is not minimal in terms of mutual information. This report aims to clarify this situation, since when we talk about entangled systems we may often only include the quantum degrees of freedom that are entangled and ignore the remaining quantum degrees of freedom. Not including all quantum degrees of freedom affects the mutual information between the subsystems and with their inclusion the mutual information is no longer maximal. The null distance or formation of wormholes between subsystems becomes an artefact of our ignorance of the full system.

3 Closing the wormholes

The main object of interest in this report is EPR pairs which are maximally entangled particles, most often described as a pair of entangled qubits. We will begin by looking at such 2-qubit EPR pairs, often called Bell states, and compute the mutual information between them. We will then generalise EPR pairs with GHZ states and then introduce the environment into the picture and how mutual information changes in all these cases. Decoherence is then introduced w.r.t. how it changes mutual information and how not only does it change the state we find the EPR pair in but also the local environment that decoheres each particle of the EPR pair.

3.1 Maximally entangled EPR pair

We start with a maximally entangled EPR pair, considering the TPS (see Appendix A.1) $\mathcal{T} : \mathcal{H} \mapsto \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, which we will as tradition dictates call Alice and Bob

$$\begin{aligned} |\psi\rangle_{AB} &= \frac{1}{\sqrt{2}} \left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right) \\ &= \frac{1}{\sqrt{2}} \Big(|00\rangle_{AB} + |11\rangle_{AB} \Big) . \end{aligned} \tag{3.1.1}$$

Being a pure state, we can write its density matrix as

$$\begin{aligned} \rho_{AB} &= |\psi\rangle\langle\psi|_{AB} \\ &= \frac{1}{2} \Big(|00\rangle\langle 00|_{AB} + |00\rangle\langle 11|_{AB} + |11\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB} \Big) . \end{aligned} \tag{3.1.2}$$

Since it is a pure state, the von Neumann entropy is 0. This is because a diagnonalisation yields a single unity eigenvalue and remaining null eigenvalues, so the joint von Neumann entropy (see Appendix A.4) S_{AB} is

$$S_{AB} = S(\rho_{AB}) = -\sum_i \lambda_i \ln \lambda_i = -\ln 1 = 0 . \tag{3.1.3}$$

To find the mutual information, we need to first find the reduced density matrices by tracing out A and B

$$\rho_A = \mathrm{tr}_B \rho_{AB} = \sum_{i=0,1} \langle i|_B \, \rho_{AB} \, |i\rangle_B = \frac{1}{2} \Big(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A \Big) \tag{3.1.4}$$

$$\rho_B = \mathrm{tr}_A \rho_{AB} = \sum_{i=0,1} \langle i|_A \, \rho_{AB} \, |i\rangle_A = \frac{1}{2} \Big(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B \Big) . \tag{3.1.5}$$

We can see from Eq. (A.3.3) that the reduced density matrices are maximally mixed since both can be written in the form

$$\rho = \frac{1}{k} \mathbb{1}^k , \tag{3.1.6}$$

and the entropy of a maximally mixed states is given by $\ln k$ where in this case $k = 2$. Shown explicitly

$$S_A = S(\rho_A) = -\sum_{i=1}^{2} \frac{1}{2} \ln \frac{1}{2} = -\ln \frac{1}{2} = \ln 2 . \tag{3.1.7}$$

The mutual information, how much we could for example learn from B by knowing A, is therefore computed to be

$$I(A:B) = S_A + S_B - S_{AB} = 2\ln 2\,, \tag{3.1.8}$$

which is maximal, as proven in Eq. (A.5.5).

Since the mutual information is maximal, if one naïvely plugged this mutual information into our metric

$$d(A,B) = l_{RC}\Phi\left(\frac{I(A\!:\!B)}{I_0}\right) = l_{RC}\Phi\left(\frac{2\ln 2}{2\ln 2}\right) = l_{RC}\Phi(1) = 0\,, \tag{3.1.9}$$

which we know cannot be true as we can create entangled pairs in a lab and separate them with a non-zero distance. Clearly, either the entire approach is misconstrued or we need to be more careful about what exactly we are computing the mutual information between.

3.2 Generalising EPR pairs with the GHZ state

We generalise the previous case of an entangled 2-qubit system by considering K entangled qudits[2] with the TPS $\mathcal{T}:\mathcal{H}\mapsto\bigotimes_k^K\mathcal{H}_k$, also known as the Greenberger–Horne–Zeilinger or GHZ state

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i_1,\ldots,i_K\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}\bigotimes_{n=1}^{K}|i_n\rangle\ . \tag{3.2.1}$$

This state can be simply written as

$$|\psi\rangle = \frac{1}{\sqrt{N}}\left(|0\ldots 0\rangle + \ldots + |N-1-1\rangle\right)\,, \tag{3.2.2}$$

where if $N=2$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\ldots 0\rangle + |1\ldots 1\rangle\right)\,, \tag{3.2.3}$$

and if $K=2$ we simply get our EPR state. The pure density matrix representing this state is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{N}\sum_{i,j=0}^{N-1}\bigotimes_{n=1}^{K}|i_n\rangle\langle j_n|\ . \tag{3.2.4}$$

Recovering the reduced density matrix ρ_1 belonging to subspace $\mathcal{H}_1$

$$\begin{aligned}\rho_1 &= \frac{1}{N}\sum_{i,j,k=0}^{N-1}|i_1\rangle\langle j_1|\bigotimes_{n=2}^{K}\langle k_n|i_n\rangle\langle j_n|k_n\rangle\\ &= \frac{1}{N}\sum_{i,j=0}^{N-1}|i_1\rangle\langle j_1|\,\delta_{ij}\\ &= \frac{1}{N}\sum_{i=0}^{N-1}|i_1\rangle\langle i_1|\ ,\end{aligned} \tag{3.2.5}$$

[2]Where a qubit has two states $\{|0\rangle, |1\rangle\}$, a qudit has d states, $\{|0\rangle,\ldots,|d\rangle\}$

and noting the symmetry of the subsequent cases $\rho_2, \ldots, \rho_K$, we can identify that in general we will get maximally mixed density matrices ρ_i in the individual subspaces $\mathcal{H}_i$ when we have a maximally entangled pure state $|\psi\rangle$ in the composite space $\mathcal{H}_{\otimes_K}$. Explicitly

$$\rho_i = \frac{1}{N} \sum_{j=0}^{N-1} |j_i\rangle\langle j_i| = \frac{1}{N} \mathbb{1}_N \,, \tag{3.2.6}$$

and so

$$S_i = S(\rho_i) = -\sum_{i=1}^{N} \frac{1}{N} \ln \frac{1}{N} = N \frac{1}{N} \ln N = \ln N \,. \tag{3.2.7}$$

For the cases $K > 2$, to find the reduced joint density matrix belonging to two subspaces $\mathcal{H}_i, \mathcal{H}_j$, we trace out the remaining degrees of freedom

$$\begin{aligned} \rho_{12} &= \frac{1}{N} \sum_{i,j,k=1}^{N} (\,|i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2|\,) \bigotimes_{n=3}^{K} \langle k_n|i_n\rangle \, \langle j_n|k_n\rangle \\ &= \frac{1}{N} \sum_{i,j=1} (\,|i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2|\,)\delta_{ij} \\ &= \frac{1}{N} \sum_{i=1}^{N} |i_1 i_2\rangle\langle i_1 i_2| \;. \end{aligned} \tag{3.2.8}$$

Once again from symmetry we can write that

$$\rho_{ij} = \frac{1}{N} \sum_{k=1}^{N} |k_i k_j\rangle\langle k_i k_j| \;, \tag{3.2.9}$$

which is a $N^2 \times N^2$ diagonal matrix representing a mixed state (but not maximally mixed) with N eigenvalues, each being $\frac{1}{N}$. We therefore compute the joint entropy as

$$S_{ij} = S(\rho_{ij}) = -\sum_{n=1}^{N} \frac{1}{N} \ln \frac{1}{N} = N \frac{\ln N}{N} = \ln N \,. \tag{3.2.10}$$

We can therefore state that the mutual information between any two N-dimensional subsystems in $\mathcal{H}_i, \mathcal{H}_j$, given that the total system is maximally entangled in $\mathcal{H}_{\otimes_K}$, is

$$I(i:j) = S_i + S_j - S_{ij} = 2\ln N - \ln N = \ln N \,. \tag{3.2.11}$$

The mutual information between two subsystems of a larger, maximally entangled system of $K > 2$ subsystems is *independent* of the number of subsystems K and depends *only* on the number of states N in each subsystem.

Suppose now that we consider composite subsystems with the goal of computing $I(1, \ldots, J : J+1, \ldots, K)$ for some $1 \leq J \leq K < L$ where the TPS for our Hilbert space is $\mathcal{T} : \mathcal{H} \mapsto \bigotimes_l^L \mathcal{H}_l$. Here we choose to label the subsystems so that they can be ordered into two composite systems given by the index J, so the composite subsystem A will consist of subsystems 1 while the composite subsystem B will consist of subsystems

$J+1$. We can generalise our result of the joint density matrix from Eq. (3.2.9) for some composite subsystem A as

$$\rho_A = \frac{1}{N}\sum_{k=1}^{N}\bigotimes_{i=1}^{J}|k_i\rangle\langle k_i| \; , \tag{3.2.12}$$

and for B as

$$\rho_B = \frac{1}{N}\sum_{k=1}^{N}\bigotimes_{i=J+1}^{K}|k_i\rangle\langle k_i| \; , \tag{3.2.13}$$

and we find equally for the joint matrix of the composites

$$\rho_{AB} = \frac{1}{N}\sum_{k=1}^{N}\bigotimes_{i=1}^{K}|k_i\rangle\langle k_i| \; . \tag{3.2.14}$$

These are $N^J \times N^J$, $N^{K-J} \times N^{K-J}$ and $N^K \times N^K$ diagonal matrices, respectively, with each eigenvalue being $\frac{1}{N}$ and there are N such eigenvalues in each. Computing the reduced entropies we find

$$\begin{aligned} S_A = S(\rho_A) &= -\sum_{i=1}^{N}\frac{1}{N}\ln\frac{1}{N} = \ln N \\ S_B = S(\rho_B) &= -\sum_{i=1}^{N}\frac{1}{N}\ln\frac{1}{N} = \ln N \, , \end{aligned} \tag{3.2.15}$$

while the joint entropy is

$$S_{AB} = S(\rho_{AB}) = -\sum_{i=1}^{N}\frac{1}{N}\ln\frac{1}{N} = \ln N \, , \tag{3.2.16}$$

meaning that the mutual information is

$$I(A\!:\!B) = \ln N \, . \tag{3.2.17}$$

Once again we find that for any arbitrary composite of subsystems of a maximally entangled system, the mutual information between any two composites where $K < L$ is independent of how many subsystems there are and is given by $\ln N$. If and only if $K = L$, that is to say we have effectively bipartitioned the entire system, then the mutual information is $2\ln N$ i.e with respect to bipartitioning there is no continuous mapping. The reason for this is that the entropy of each composite is $\ln N$, independent of the size of the partition, as long as the size of the partition is less than the size of the total system L. The joint entropy on the other hand can be either $\ln N$ if its size is less than the size of the system L or 0 if it is the size of the whole system L. As soon as all subsystems are included i.e. $K = L$, the joint entropy is 0 and the mutual information is $\ln N$ instead of $2\ln 2$.

It is important to stress the following: if a system is *maximally entangled* in a GHZ-state, we must have access to *all* subsystems for the system to look entangled, otherwise it appears to be in a mixture. If our system is very large, even removing a single subsystem removes any information of entanglement in the system. The system appears to be maximally entangled with the relative entropy of $2\ln N$ only once we include all degrees of freedom. This is a rather extreme example of how important it is for an emergent spacetime approach based on mutual information to include all quantum degrees of freedom.

3.3 EPR pair with environment

Let us now consider our EPR pair AB together with a rudimentary environment C which we model as a single qubit with a coefficient parametrisation w.r.t. α to ensure the state is normalised

$$|e\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle\ . \tag{3.3.1}$$

Defining the product state

$$\begin{aligned} |\Psi\rangle &= |\psi\rangle_{AB} \otimes |e\rangle_C \\ &= \frac{1}{\sqrt{2}}(\,|00\rangle + |11\rangle\,)(\,\cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle\,) \\ &= \frac{1}{\sqrt{2}}\Big(\cos\alpha(\,|000\rangle + |110\rangle\,) + \sin\alpha(\,|111\rangle + |001\rangle\,)\Big)\,, \end{aligned} \tag{3.3.2}$$

we compute the density matrix for the product state ABC

$$\begin{aligned} \rho_{ABC} = \frac{1}{2}\Big(&\cos^2\alpha(\,|000\rangle\langle 000| + |000\rangle\langle 110| + |110\rangle\langle 000| + |110\rangle\langle 110|\,) \\ +\ \ &\cos\alpha\sin\alpha(\,|000\rangle\langle 111| + |000\rangle\langle 001| + |110\rangle\langle 111| + |110\rangle\langle 001|\,) \\ +\ \ &\cos\alpha\sin\alpha(\,|111\rangle\langle 000| + |111\rangle\langle 110| + |001\rangle\langle 000| + |001\rangle\langle 110|\,) \\ +\ \ &\sin^2\alpha(\,|111\rangle\langle 111| + |111\rangle\langle 001| + |001\rangle\langle 111| + |001\rangle\langle 001|\,)\Big)\,. \end{aligned} \tag{3.3.3}$$

We can then recover our density matrix ρ_{AB} by tracing out the environment

$$\begin{aligned} \rho_{AB} &= \mathrm{tr}_C \rho_{ABC} \\ &= \sum_{i=0,1} (\mathbb{1}_{AB} \otimes \langle i|_C)\rho_{ABC}(\mathbb{1}_{AB} \otimes |i\rangle_C) \\ &= \frac{1}{2}\Big(\,|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\,\Big)\,. \end{aligned} \tag{3.3.4}$$

Tracing out A and B instead

$$\begin{aligned} \rho_C &= \mathrm{tr}_{AB}\rho_{ABC} \\ &= \sum_{i,j=0,1} \langle i_A j_B|\rho_{ABC}|i_A j_B\rangle \\ &= \Big(\cos^2\alpha\,|0\rangle\langle 0| + \cos\alpha\sin\alpha(\,|0\rangle\langle 1| + |1\rangle\langle 0|\,) + \sin^2\alpha\,|1\rangle\langle 1|\,\Big)\,, \end{aligned} \tag{3.3.5}$$

which we see represents the pure state

$$\rho_C = |e\rangle\langle e|\ . \tag{3.3.6}$$

Now we trace out only B

$$\begin{aligned}\rho_{AC} &= \mathrm{tr}_B \rho_{ABC} \\ &= \frac{1}{2}\Big(\cos^2\alpha(|00\rangle\langle 00| + |10\rangle\langle 10|) \\ &\quad + \cos\alpha \sin\alpha(|00\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| + |01\rangle\langle 00|) \\ &\quad + \sin^2\alpha(|11\rangle\langle 11| + |01\rangle\langle 01|)\Big) ,\end{aligned} \tag{3.3.7}$$

and then only A

$$\rho_{BC} = \mathrm{tr}_A \rho_{ABC} = \rho_{AC} \,. \tag{3.3.8}$$

Diagonalising these matrices yields the characteristic polynomial

$$\lambda^2 \left(\lambda^2 - \lambda + \frac{1}{4} \right) = 0 \,, \tag{3.3.9}$$

which has the solutions $\lambda = 0, 0, \frac{1}{2}, \frac{1}{2}$. In other words, ρ_{AC} and ρ_{BC} represent mixed states with the entropy

$$S_{AC} = S_{BC} = -\left(\frac{1}{2}\ln\frac{1}{2} + \frac{1}{2}\ln\frac{1}{2} \right) = \ln 2 \,. \tag{3.3.10}$$

We can then conclude that for the pure states we have found, $S_{ABC} = S_{AB} = S_C = 0$ and as in our previous example, $S_A = S_B = \ln 2$. Finally, we compute the joint entropy

$$S(AC, BC) = -tr[\rho_{AC}] \,. \tag{3.3.11}$$

The combinations of mutual information we can form are then

$$\begin{aligned} I(A\!:\!B) &= S_A + S_B - S_{AB} = 2\ln 2 \\ I(A\!:\!C) &= S_A + S_C - S_{AC} = 0 \\ I(B\!:\!C) &= S_B + S_C - S_{BC} = 0 \\ I(AB\!:\!C) &= S_{AB} + S_C - S_{ABC} = 0 \\ I(AC\!:\!B) &= S_{AC} + S_B - S_{ABC} = 2\ln 2 \\ I(A\!:\!BC) &= S_A + S_{BC} - S_{ABC} = 2\ln 2 \,. \end{aligned} \tag{3.3.12}$$

To interpret these results, $I(A:C) = I(B:C) = I(AB:C) = 0$ tells us that the environment C is entirely disentangled or independent from our EPR pair and so no amount of information we acquire from the environment C will inform us about either A, B or AB. Conversely, $I(A\!:\!B) = I(AC\!:\!B) = I(A\!:\!BC) = 2\ln 2$ tells us that whatever information we acquire about B will inform us maximally about the information of A and vice versa, independent of whatever additional information we acquire from the environment C together with either A or B. Therefore, no additional information is gained about A by knowing BC compared to only knowing B and the same for B by knowing AC compared to only knowing A.

3.4 EPR pair entangled with an environment

Let us now suppose there is a unitary transformation U such that we entangle our EPR pair AB with the environment C

$$|\psi\rangle \otimes |e\rangle \overset{U}{\mapsto} |\psi'\rangle = \frac{1}{\sqrt{2}}(\,|000\rangle + |111\rangle\,) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \bigotimes_{n=1}^{K} |i_n\rangle\,, \quad N = 2, K = 3\,. \tag{3.4.1}$$

We define the density matrix

$$\rho_{ABC} = \frac{1}{N} \sum_{i,j=0}^{N-1} \bigotimes_{n=1}^{K} |i_n\rangle\langle j_n|\;, \tag{3.4.2}$$

which is a pure state so $S_{ABC} = 0$. Now finding the reduced density matrices by performing the respective traces and noting the symmetry $\rho_A = \rho_B = \rho_C$ and $\rho_{AB} = \rho_{BC} = \rho_{AC}$

$$\begin{aligned} \rho_A &= \mathrm{tr}_{BC}\rho_{ABC} = \sum_{i,j=0,1} \langle i_B j_C|\rho_{ABC}|i_B j_C\rangle = \frac{1}{2}(\,|0\rangle\langle 0| + |1\rangle\langle 1|\,) \\ \rho_{AB} &= \mathrm{tr}_C\rho_{ABC} = \sum_{i=0,1} \langle i_C|\rho_{ABC}|i_C\rangle = \frac{1}{2}(\,|00\rangle\langle 00| + |11\rangle\langle 11|\,)\,. \end{aligned} \tag{3.4.3}$$

The reduced density matrices ρ_A,ρ_B,ρ_C are all maximally mixed, while ρ_{AC},ρ_{BC},ρ_{AB} are mixed with eigenvalues $\lambda = 0, 0, \frac{1}{2}, \frac{1}{2}$, so the corresponding entropies and joint entropies are

$$\begin{aligned} S_A = S_B = S_C &= \ln 2 \\ S_{AC} = S_{BC} = S_{AB} &= \ln 2\,. \end{aligned} \tag{3.4.4}$$

The combinations of mutual information we can now form are

$$\begin{aligned} I(A:B) &= S_A + S_B - S_{AB} = \ln 2 \\ I(A:C) &= S_A + S_C - S_{AC} = \ln 2 \\ I(B:C) &= S_B + S_C - S_{BC} = \ln 2 \\ I(AB:C) &= S_{AB} + S_C - S_{ABC} = 2\ln 2 \\ I(AC:B) &= S_{AC} + S_B - S_{ABC} = 2\ln 2 \\ I(A:BC) &= S_A + S_{BC} - S_{ABC} = 2\ln 2\,. \end{aligned} \tag{3.4.5}$$

To interpret our findings, if we entangle AB with the environment C, then Bob cannot gain maximal information about Alice by only having access to his own information. Likewise, Alice or Bob cannot gain maximal information about the environment C by only having access to their own respective information. If on the other hand Alice and Bob share their information, they may now gain maximal information about the environment, and likewise if Alice or Bob also have information about the environment C, they can gain maximal information about each other. Once they have gained this additional information, no more information can be gained about further joining of states.

Comparing this to the case when the environment C was *not* entangled with AB, the mutual information between A and B is now smaller. In other words, once AB entangles with an environment C, the mutual information between Alice and Bob decreases. This can be understood in terms of the **Monogamy of Entanglement** (See Appendix A.7): if A is entangled with B, then entangling A with C corresponds to a disentangling of A with B, and conversely, entangling B with C corresponds to a disentangling of B with A.

3.5 Decoherence of an entangled pair and the loss of information

So far we have modelled the environment in the most rudimentary of ways. We have also seen how tracing out degrees of freedom of our state can have a drastic impact on mutual information. We will now generalise the idea in Sec. 3.4 and see what happens to the mutual information as we trace the environment out.

We consider our pair AB in an initially pure state, written as a Schmidt decomposition

$$|\psi\rangle = \sum_I \alpha_I \, |u_I\rangle_A \otimes |v_I\rangle_B \ , \tag{3.5.1}$$

and our environment is also in some much larger pure state $|\varepsilon\rangle$. Together they initially form the tensor product state

$$\begin{aligned} |\Psi\rangle &= |\psi\rangle \otimes |\varepsilon\rangle \\ &= \sum_{IJ} \alpha_I \varepsilon_J \, |u_I\rangle_A \otimes |v_I\rangle_B \otimes |\varepsilon_J\rangle \\ &= \sum_{IJ} \alpha_I \varepsilon_J \, |u_I v_I \varepsilon_J\rangle \ . \end{aligned} \tag{3.5.2}$$

The initial density operator can then be written as

$$\rho_{AB\mathcal{E}} = |\psi\rangle\langle\psi| \otimes |\varepsilon\rangle\langle\varepsilon| \ . \tag{3.5.3}$$

Tracing out the environment yields the expected density operator

$$\begin{aligned} \rho_{AB} &= \mathrm{tr}_{\mathcal{E}} \rho_{AB\mathcal{E}} \\ &= |\psi\rangle\langle\psi| \\ &= \sum_{IJ} \alpha_I \alpha_J^* \, |u_I v_I\rangle\langle u_J v_J| \ , \end{aligned} \tag{3.5.4}$$

whose entropy by virtue of being a pure state is $S(\rho_{AB}) = 0$. Tracing out B

$$\begin{aligned} \rho_A &= \mathrm{tr}_B \rho_{AB} \\ &= \sum_{IJK} \alpha_I \alpha_J^* \, \langle v_K | u_I v_I\rangle \ \langle u_J v_J | v_K\rangle \\ &= \sum_I |\alpha_I|^2 |u_I\rangle\langle u_I| \ , \end{aligned} \tag{3.5.5}$$

meaning that the entropy of the reduced states are

$$S(\rho_A) = -\sum_I |\alpha_I|^2 \ln |\alpha_I|^2 = S(\rho_B) \, , \tag{3.5.6}$$

and so the mutual information is

$$I(A\!:\!B) = -2 \sum_I |\alpha_I|^2 \ln |\alpha_I|^2 \, . \tag{3.5.7}$$

We then consider a unitary transformation U given by some Hamiltonian such that

$$\begin{aligned} U : |\Psi\rangle \mapsto |\Psi'\rangle &= \sum_I \beta_I \, |u_I\rangle_A \otimes |v_I\rangle_B \otimes |\varepsilon_I\rangle \\ &= \sum_I \beta_I \, |u_I v_I \varepsilon_I\rangle \ , \end{aligned} \tag{3.5.8}$$

i.e. entangling the environment $\mathcal{E}$ with the the pair AB for some coefficients β_I. Tracing out the environment in the basis of the environment and invoking the einselection criterion that $\langle\varepsilon_i|\varepsilon_j\rangle = \delta_{ij}$ [24] yields

$$\begin{aligned} \rho'_{AB} &= \mathrm{tr}_{\mathcal{E}} \rho'_{AB\mathcal{E}} \\ &= \sum_{IJK} \beta_I \beta_J^* \, |u_I v_I\rangle\langle u_J v_J| \otimes \langle\varepsilon_K|\varepsilon_I\rangle \, \langle\varepsilon_J|\varepsilon_K\rangle \\ &= \sum_{IJ} \beta_I \beta_J^* \delta_{IJ} \, |u_I v_I\rangle\langle u_J v_J| \\ &= \sum_I |\beta_I|^2 |u_I v_I\rangle\langle u_I v_I| \ , \end{aligned} \tag{3.5.9}$$

which is a mixed, separable state with non-zero joint entropy equal to the reduced entropies

$$S(\rho'_{AB}) = S(\rho'_A) = S(\rho'_B) = -\sum_I |\beta_I|^2 \mathrm{ln} |\beta_I|^2 \, , \tag{3.5.10}$$

meaning that the mutual information is

$$I'(A\!:\!B) = -\sum_I |\beta_I|^2 \mathrm{ln} \, |\beta_I|^2 \, . \tag{3.5.11}$$

i.e. at most classically correlated since for classical variables X, Y the mutual information is bounded by the individual entropy $I(X,Y) \leq H(X)$. This can be seen by defining mutual information in terms of conditional entropy, $I(X\!:\!Y) = H(X) - H(X|Y)$ where classically $H(X|Y) \geq 0$.

The change in entropy of the system, or the change in information available to AB, is

$$\Delta S = S(\rho'_{AB}) - S(\rho_{AB}) = -\sum_I |\beta_I|^2 \mathrm{ln} |\beta_I|^2 \, , \tag{3.5.12}$$

i.e. entropy has increased in the system corresponding to a loss of information to the environment so

$$S(\rho'_{\mathcal{E}}) = \Delta S \, . \tag{3.5.13}$$

The change in mutual information is

$$\begin{aligned} \Delta I(A\!:\!B) &= I'(A\!:\!B) - I(A\!:\!B) \\ &= -\sum_I \left(|\beta_I|^2 \mathrm{ln} |\beta_I|^2 - 2|\alpha_I|^2 \mathrm{ln} |\alpha_I|^2 \right) \ , \end{aligned} \tag{3.5.14}$$

where $\Delta I(A:B) \leq 0$ since mutual information can only decrease under a quantum channel [25]. Writing it in terms of entropy

$$\begin{aligned}\Delta I(A\!:\!B) &= (S(\rho'_A) + S(\rho'_B) - S(\rho'_{AB})) - (S(\rho_A) + S(\rho_B) - S(\rho_{AB})) \\ &= S(\rho'_A) - 2S(\rho_A)\,,\end{aligned} \tag{3.5.15}$$

and letting the initial state be a GHZ-state, the change in mutual information is equal to the negative change in entropy

$$\Delta I(A\!:\!B)_{GHZ} = -\Delta S_{GHZ} = -\ln N\,. \tag{3.5.16}$$

The process of decoherence w.r.t. the pair AB involves entangling the pair AB with an environment $\mathcal{E}$ such that when the environment is traced out, all off-diagonal terms of the density matrix for the pair AB vanish i.e information has been lost to the environment. For a GHZ state, the change in mutual information of the system corresponds exactly to the change of entropy in the environment, which is $\ln N$.

3.6 Mutual information of an EPR pair in the momentum sector

In the original paper by Einstein, Podolsky and Rosen [26], they consider a pair of particles entangled in momentum and position. Later, Bohm and Aharonov [27] then considered a pair of particles initially entangled in spin then separated, the type of EPR pair later considered by Bell [28]. We will combine the two and consider a spin-0 particle decay into an EPR pair, conserving spin and momentum, as a pure state. To conserve spin and momentum a state must then have opposite spins for each subsystem and momenta of opposite but equal magnitude. There is no reason to prefer any particular configuration though, in an Everettian sense each branch is equally possible, so we consider a linear combination of all possible combinations of spins and momenta

$$|\psi\rangle = a\,|0, p_A, 1, -p_B\rangle + b\,|0, -p_A, 1, p_B\rangle + c\,|1, p_A, 0, -p_B\rangle + d\,|1, -p_A, 0, p_B\rangle\ , \tag{3.6.1}$$

where a, b, c, d are commuting coefficients. In other words, we consider two particles with opposite momenta and opposite spins with the total spin and momentum of system, respectively, being 0 i.e. conserved. We assume that the spin degrees of freedom and momentum degrees of freedom are separable as

$$\rho_{AB} = \rho^s_{AB} \otimes \rho^p_{AB}\,, \tag{3.6.2}$$

which is equivalent to saying that the total state is a tensor product of a spin state and a momentum state. When we compute the mutual information of an EPR pair, we generally compute this for only the spin degrees of freedom

$$I(A^s\!:\!B^s) = S(\rho^s_A) + S(\rho^s_B) - S(\rho^s_{AB})\,, \tag{3.6.3}$$

but suppose we include the momentum degrees of freedom

$$\begin{aligned}I(A:B) &= S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \\ &= S(\rho^s_A \otimes \rho^p_A) + S(\rho^s_B \otimes \rho^p_B) - S(\rho^s_{AB} \otimes \rho^p_{AB}) \\ &= S(\rho^s_A) + S(\rho^p_A) + S(\rho^s_B) + S(\rho^p_B) - S(\rho^s_{AB}) - S(\rho^p_{AB}) \\ &= I(A^s:B^s) + I(A^p:B^p)\,,\end{aligned} \tag{3.6.4}$$

which trivially implies $I(A:B) \geq I(A^s:B^s)$.

It is then immediately clear that one may have an entangled EPR pair whose total mutual information exceeds the spin mutual information i.e. it is not always maximal. Qualitatively, we then consider that initially an entangled EPR pair has maximal mutual information both in spin and momentum and as we separate them, the mutual information in the momentum sector decreases. It also then becomes clear that when the size of the Hilbert space for the momentum sector far exceeds the size of the Hilbert space for the spin sector, the mutual information of the spin sector is negligible in comparison and so

$$I(A{:}B) \approx I(A^p{:}\,B^p)\,, \tag{3.6.5}$$

and correspondingly the metric of an EPR pair would be largely independent of spin correlation

$$d(A:B) \sim -\log \frac{I(A^p{:}\,B^p)}{I_0^p}\,. \tag{3.6.6}$$

3.7 EPR pairs decohering in the momentum sector

We will now consider a more realistic example of an EPR pair with both a spin and momentum state. There is reason to believe the Hilbert space of a quantum gravity theory is finite dimensional [29]. If so, one would need discrete momentum and position operators, which can be achieved by Generalised Pauli Operators [30]. This topic is explored further in Sec. 4.2.

We imagine the decay of a particle $M \to m + m$ and suppose the momentum of the initial particle is $p + \Delta p$. Since we expect that we can localise our particles in a lab of scale l_{lab}, we suppose the momentum uncertainty is upper bounded by $\hbar/l_{lab}$. We further assume that it must be lower bounded by some IR scale, l_{IR}, which we infer is on the scale of the cosmological constant Λ such that $l_{IR} \sim \Lambda^{-\frac{1}{2}} \sim 10^{26}$ m. We suppose therefore that the lower bound is given by $p_{IR} = \hbar\Lambda^{\frac{1}{2}}$ which is on the order 10^{-60} kg·m/s. In other words, we bound the momentum uncertainty by

$$p_{IR} \leq \Delta p \leq p_{lab} = \frac{\hbar}{l_{lab}}\,. \tag{3.7.1}$$

Conserving momentum we get

$$p + \Delta p = p_1 + \Delta p_1 + p_2 + \Delta p_2\,. \tag{3.7.2}$$

In the rest frame of M so that $p = 0$, conserving momentum for the two daughter particles, $p_1 = -p_2$ we get

$$\Delta p_1 + \Delta p_2 = \Delta p\,, \tag{3.7.3}$$

meaning we express the second momentum in terms of the first as $\Delta p_2 = \Delta p - \Delta p_1$. If we are in the rest frame on the other hand, we are minimizing our uncertainty in position for M and so the uncertainty Δp associated with M must be maximal, so $\Delta p = p_{lab}$.

The bounds on Δp_1 must just like Δp satisfy the lower bound given by p_{IR} and the upper bound given by p_{lab} so

$$p_{IR} \leq \Delta p_1 \leq p_{lab}\,. \tag{3.7.4}$$

Given expectation values p_1, p_2 for the momenta of the particles, we suppose we can express the momentum state as linear combinations of all possible momentum states w.r.t. all possible uncertainties, summing from p_{IR} to mc in steps of of p_{IR}[3]

$$|\pi\rangle = \sum_{\Delta p_1 = p_{IR}}^{p_{lab}-p_{IR}} \alpha_{\Delta p_1} |p_1 + \Delta p_1, p_{lab} - (p_1 + \Delta p_1)\rangle \ , \tag{3.7.5}$$

where $\alpha_{\Delta p_1}$ determines the distribution of the states.

Since the momentum state here only really depends on the uncertainty, we will omit p_1, writing now with the bounds changed

$$|\pi\rangle = \sum_{n=1}^{p_{lab}/p_{IR}-1} \alpha_n |np_{IR}, p_{lab} - np_{IR}\rangle \ , \tag{3.7.6}$$

and redefining the bounds so that $N = p_{lab}/p_{IR}$

$$|\pi\rangle = \sum_{n=1}^{N-1} \alpha_n |n, N-n\rangle \ , \tag{3.7.7}$$

Taking the density operator

$$\rho^p = \sum_{i,j=1}^{N-1} \alpha_i \alpha_j^* |i, N-i\rangle\langle j, N-j| \ . \tag{3.7.8}$$

we compute the reduced density operator

$$\begin{aligned}
\rho^p &= \sum_{i,j,k=1}^{N-1} \alpha_i \alpha_j^* |i\rangle\langle j| \otimes \langle k, N-i\rangle \langle N-j, k\rangle \\
&= \sum_{i,j,k=1}^{N-1} \alpha_i \alpha_j^* \delta_{k,N-i}, \delta_{k,N-j} |i\rangle\langle j| \\
&= \sum_{i=1}^{N-1} |\alpha_i|^2 |i\rangle\langle i| \ ,
\end{aligned} \tag{3.7.9}$$

and so the mutual information in the momentum sector is given by

$$I(A^p\!:\!B^p) = -2\sum_{i=1}^{N-1} |\alpha_i|^2 \ln|\alpha_i|^2 \,. \tag{3.7.10}$$

Assuming an even distribution $|\alpha_i|^2 = \frac{1}{N-1}$, explicitly we compute it in base 10, given that $N = l_{IR}/l_{lab}$

$$I(A^p\!:\!B^p) = 2\log(N-1) \approx 2\log N = 2(\log l_{IR} - \log l_{lab}) \approx 2(26 - \log l_{lab}) \,. \tag{3.7.11}$$

We interpret here l_{lab} as the scale of decoherence i.e. a longer l_{lab} implies a smaller environment with which the system decoheres and correspondigly the mutual information

[3]The difference in p_{IR} and mc is so large that even though c is not an integer multiple of q_{IR}, an integer multiple of q_{IR} would bring us close enough to effectively be true.

between the pair remains high. A longer l_{lab} implies a larger environment and correspondingly mutual information between the pair is smaller.

When the EPR pair separates, it is this initial mutual information in the momentum sector that decreases as the momentum degrees of freedom decohere w.r.t. the environment. Taking the ratio of mutual information in spin to momentum

$$\frac{I(A^s : B^s)}{I(A^p : B^p)} \approx \frac{\log 2}{26 - \log l_{lab}} \, . \tag{3.7.12}$$

By taking the Planck scale as a lower bound for l_{lab}, the ratio of mutual information is lower bounded by

$$10^{-3} \lessapprox \frac{I(A^s : B^s)}{I(A^p : B^p)} \, . \tag{3.7.13}$$

For a metric based on $-\log(i)$ where $i = I(A:B)/I_0$, by varying the mutual information of spin between A and B so that $i \to i + \Delta i$ and taking the ratio $\log(i + \Delta i)/\log(i)$, as $i \to 1$, the function diverges infinitely for $\Delta i > 0$ compared to $\Delta i < 0$. Our construction above gives an experimental opportunity for testing spacetime emergence.

3.8 Decoherence of an entangled pair with an entangled environment

We imagine a thought experiment, where we start with an EPR pair and disentangle it while entangling each part with the environment. Let there be a Hilbert space $\mathcal{H}$ with a TPS $\mathcal{T}$ such that

$$\mathcal{T} : \mathcal{H} \mapsto \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \bigotimes_n \left(\mathcal{H}_A^n \otimes \mathcal{H}_B^n\right) \, , \tag{3.8.1}$$

where n represents sectors and A,B represent subsystems. Let there be a bipartite density matrix $\rho_{AB} : \mathcal{H}_{AB} \mapsto \mathcal{H}_{AB}$ for the subsystems A and B, initially pure, that can be separated into n different sectors

$$\rho_{AB} = \bigotimes_n \rho_{AB}^n \, . \tag{3.8.2}$$

We assume for now that any interaction Hamiltonian which we unitarily transform states with has no interaction between sectors, only between subsystems of sectors. We introduce an environment $\mathcal{E}$ with the density matrix $\rho_{\mathcal{E}}$. From here on we use greek symbols for environmental partitions and latin symbols for the system partitions. The environment takes on the same sector-wise partitioning

$$\rho_{\mathcal{E}} = \bigotimes_n \rho_{\mathcal{E}}^n \, , \tag{3.8.3}$$

where initially

$$\begin{aligned} \exists n &: \rho_{AB}^n \neq \rho_A^n \otimes \rho_B^n \\ \forall n &: \rho_{AB\mathcal{E}}^n = \rho_{AB}^n \otimes \rho_{\mathcal{E}}^n \, , \end{aligned} \tag{3.8.4}$$

or in other words, A and B are initially entangled in at least one sector, while AB and $\mathcal{E}$ are initially unentangled in each sector. For now, the environment $\mathcal{E}$ takes trivial partitions with respect to subsystems i.e. there is no non-trivial TPS $\mathcal{T}$ with respect to subsystems as of yet, so trivially we could write without loss of generality

$$\mathcal{T} : \mathcal{H}_{\mathcal{E}} \mapsto \bigotimes_n \bigotimes_\sigma \mathcal{H}^n_\sigma \,. \tag{3.8.5}$$

We consider the decoherence of AB w.r.t. the environment $\mathcal{E}$ which we have seen in Sec. 3.5 is the statement that given some time T, an entangled state ρ_{AB} when decohered transforms to a mixed state which we write using Schmidt decomposition

$$\rho_{AB} \mapsto \rho'_{AB} = \sum_i |\beta_i|^2 P^i_A \otimes P^i_B \,, \tag{3.8.6}$$

which is a fully separable state where A and B can be at most classically correlated. For the full system we correspondingly find we cannot write it as a simple tensor product such that given sufficient time

$$\rho'_{AB\mathcal{E}} \neq \rho'_{AB} \otimes \rho'_{\mathcal{E}} \,, \tag{3.8.7}$$

which is to say that we disentangle AB and entangle A and B with $\mathcal{E}$.

Let us now consider the case where we start with an entangled pair ρ_{AB} and allowing for a very long time to pass we separate them to opposite ends of the universe. It would seem reasonable to suppose that each particle now has its own separate environment α and β which the particles have entangled with, and the environments are mutually uncorrelated by virtue of being on opposite ends of the universe. In an ideal case we may consider this as the unitary transformation such that

$$U : \rho_{AB} \otimes \rho_{\mathcal{E}} \mapsto \rho_{A\alpha} \otimes \rho_{B\beta} \,. \tag{3.8.8}$$

Assuming first that AB and $\mathcal{E}$ are of the same size, we know from the monogamy of entanglement that A cannot entangle with the same degrees of freedom of $\mathcal{E}$ as B, so in the limit we introduce the bipartitioning of the environment into α and β to which A and B, respectively, entangle with. In the limit we may consider entangling system AB with the environment $\mathcal{E}$ through a unitary transformation U which induces a non-trivial partitioning of the environment into α and β by means of the TPS given by

$$\mathcal{T} : \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{\mathcal{E}} \mapsto \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \,, \tag{3.8.9}$$

which we can alternatively write as

$$\mathcal{T} : \mathcal{H}_{AB} \otimes \mathcal{H}_{\mathcal{E}} \mapsto \mathcal{H}_{A\alpha} \otimes \mathcal{H}_{B\beta} \,. \tag{3.8.10}$$

Diagrammatically it can be represented as Fig. 2.

Suppose we now consider an environment $\mathcal{E}$ much larger than AB. As AB disentangles, A and B will, respectively, only entangle with some of the degrees of freedom of $\mathcal{E}$, allowing us to describe the environment after the transformation in terms of α, β and some new $\mathcal{E}'$. Borrowing the language from Van Raamsdonk, the partitions α and β "pinch off" from $\mathcal{E}'$ [12]. Symbolically, the transformation U takes

$$U : \rho_{AB} \otimes \rho_{\mathcal{E}} \mapsto \rho_{A\alpha} \otimes \rho_{B\beta} \otimes \rho_{\mathcal{E}'} \,, \tag{3.8.11}$$

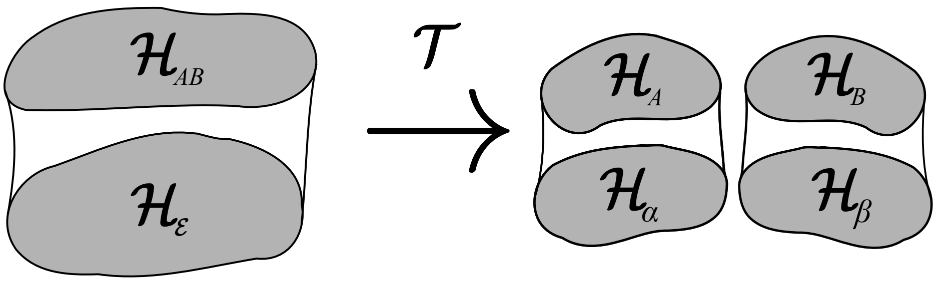

Figure 2: Disentangling of an entangled pair AB which in turn entangle with an environment $\mathcal{E}$ leading to the non-trivial partitioning of the environment into α and β

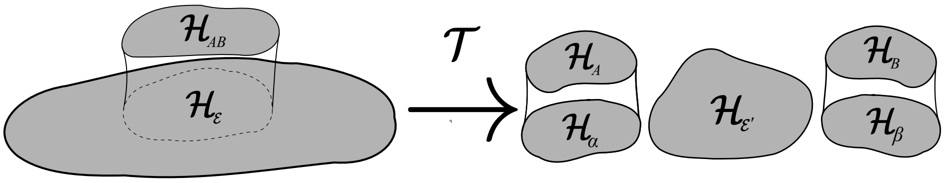

Figure 3: Disentangling of an entangled pair AB which in turn entangle with an environment ϵ in the limit leading to the "pinching off" of α and β from $\mathcal{E}'$

which diagrammatically can be seen represented in Fig. 3 While the TPS is such that

$$\mathcal{T} : \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_\mathcal{E} \mapsto \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_\alpha \otimes \mathcal{H}_\beta \otimes \mathcal{H}_{\mathcal{E}'} \,. \tag{3.8.12}$$

Let us now consider the case where we introduce the spin sector s such that our entangled pair can be written as

$$\rho_{AB} = \rho^s_{AB} \bigotimes_n \rho^n_{AB} \,, \tag{3.8.13}$$

and let the interaction Hamiltonian producing the unitary transformation U leave the spin sector relatively untouched. We can imagine this as preparing a spin entangled pair which we separate by a distance, maintaining coherence in the spin degrees of freedom but decohering the momentum degrees of freedom. As we decohere AB w.r.t. the environment $\mathcal{E}$, we again get the partitioning of the environment into α and β for the n sectors, but the entanglement between A and B in the spin degree of freedom is left untouched. In Fig. 4, we no longer consider the limit where complete disentanglement occurs, but rather wish to represent the decoherence of the momentum sector such that we end up with a mixture with non-zero mutual information between the subsystems. Diagrammatically we would write this as

$$U : \rho_{AB} \otimes \rho_\mathcal{E} \mapsto \rho^s_{AB} \otimes \sum_{ij} p_{ij} P^i_A \otimes P^i_B \,, \otimes P^j_\mathcal{E} \tag{3.8.14}$$

and once again we see that α and β are "pinching off" from $\mathcal{E}'$ while the spin entanglement between A and B connects the systems with an apparent wormhole.

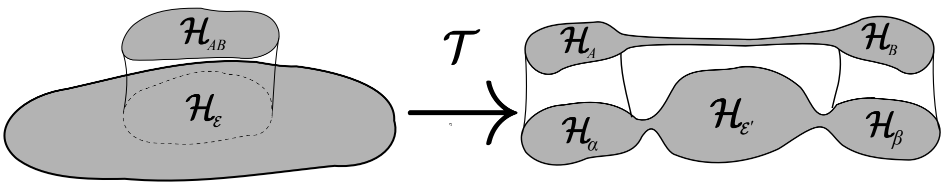

Figure 4: Decoherence of an entangled pair AB with an environment $\mathcal{E}$ with the spin-sector untouched. In this example, the limit of complete entangling of the environmental subsystems α and β with A and B has not yet occurred.

4 Discussion

When including the environment, a Hamiltonian which decoheres the momentum degrees of freedom of an EPR pair induces a partitioning of the environment, disentangling two environment regions local to each part of the pair from each other. The change in mutual information between the EPR pair can be related to the change in entropy of the environment and with the Hilbert space of the environment being much larger than that of the particles, the mutual information between the two partitions of the environment would not change appreciably as they decohere. Consequently one would not expect a noticeable distortion of space as a “small” EPR pair is separated. Conversely, when one considers systems whose size of Hilbert space approaches that of their environment, such as black holes, the change in mutual information would lead to a non-negligible change in distance between the two regions which can be interpreted as a distortion of space.

A problem that was glossed over in Sec. 3.8 is the problem of localisation. In particular, when considering an EPR pair AB separated to two distant environments α and β, we associate A with α and B with β by means of looking how entangled the particles are with their environment. A is not in β because A is entangled with α and not β. The problem arises when we reconsider the initial setup with the same line of reasoning, when AB was completely disentangled from the environment $\mathcal{E}$: where is the particle located at this point? In other words, in this framework, what does it mean to localise a particle to a particular environment? This question is left open for future research.

In collaboration with Guilherme Franzmann and Matthew Lawson, a few experimental setups have been proposed to test spacetime emergence [31]. One proposal involves the hyperpolarisation of large volumes of liquid, for example through ParaHydrogen-Induced Polarization (PHIP), Signal Amplification by Reversible Exchange (SABRE), or Xenon hyperpolarization by spin exchange with optically pumped Rubidium atoms. By varying the entanglement between the spin ensembles with NMR pulse sequences, one can with an interferometer look for variations in relative distance between the spin ensembles. Another suggestion is using a very sensitive interferometer such as the MAGIS-100 interferometer and comparing the relative distance between two populations of atoms, such as a Bose-Einstein condensate, in either entangled or unentangled states.

4.1 Factorisation of Hilbert space

In all the above computations, assumptions have been made about how to factorise a Hilbert space. We say that one part of a Hilbert space is our system and another part is our environment, but so far this seemingly innocuous assumption can only be justified to the extent that it helps us intuit a problem. The choice of factorisation becomes particularly important for an attempt at discovering any emergent geometry from mutual information. The reason is that even though von Neumann entropy is invariant w.r.t. *local* transformations, it is not invariant w.r.t. *global* transformations, i.e. the chosen TPS. Put simply, if in one TPS we have two entangled subsystems with near maximal mutual information i.e. some “short” distance, there exists another TPS in which we no longer have any entanglement, and the mutual information is 0 i.e. a “long” distance between them. More specifically, if we consider a simple Bell pair under the transformation of the global Hilbert space such that

$$\frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) \mapsto |\tilde{0}\tilde{0}\rangle_{\tilde{A}\tilde{B}} \ , \tag{4.1.1}$$

then the mutual information between the respective subsystems are $I(A:B) = 2\ln 2$ which is maximal while $I(\tilde{A}:\tilde{B}) = 0$ is minimal.

The choice of TPS, how to factorise Hilbert space, is then clearly non-trivial when considering emergent geometry and should be derived not from pedagogical intuition but from things considered more fundamental: Hilbert space, the wave function, the Schrödinger equation and some Hamiltonian. In Sec. 3.8 an argument from the Monogamy of Entanglement was given of how a Hamiltonian could induce a preferred factorisation of the environment into two separate environmental subsystems, but we still start by assuming a separation of subsystems and environment a priori. Cotler et al. [32] show that for a given Hamiltonian, if one defines a notion of a local Hamiltonian[4], then there exists a unique TPS associated with that Hamiltonian. This is extended even further to the spectrum of the Hamiltonian sufficing to define a unique TPS. The approach relies on the assumption of locality on the other hand which although intuitively reasonable still requires proper justification.

Carroll and Singh [33] produced a Quantum Mereology Algorithm for finding the bipartite factorisation that minimizes a quantity they call Schwinger entropy defined as

$$\ddot{S}_{Schwinger} = \max\left(\ddot{S}_{lin}(0), \ddot{S}_{pointer}(0)\right) \ . \tag{4.1.2}$$

$\ddot{S}_{lin}$ is the linear entanglement entropy of either of the partitions[5] and $\ddot{S}_{pointer}$ is the pointer entropy, the second order Tsallis entropy of one of the subsystems defined in the basis of an observable $\mathcal{O}$ which satisfies the Zurek commutativity criterion for pointer states, $[H_{int}, \mathcal{O}] \approx 0$. Pointer states, as introduced by Zurek, are quantum states that are left relatively invariant during decoherence i.e. unitary transformations of the system with the environment leaves particular states invariant, for example the spin degrees of freedom in Fig. 4 or in general the spin state of a Bell particle pair before measurement. The approach, although providing a possible experimental approach, is not yet theoretically satisfying as it begs the question of how such a procedure occurs in nature.

Zanardi et al. [34] show that a TPS is induced by the available interactions and operations corresponding to the set of observables "accessible" to the system w.r.t a Hamiltonian. By expressing the irreducible representations of the observables, it is then possible to form a hierarchy of TPS's where the "natural" one is one in which all observables are irreducible. In other words they argue that a preferred TPS depends on the Hamiltonian of the system and that it makes little physical sense to speak of a TPS without also including the set of observables acting on the system. In light of this, Cotler et al. [32] defined an equivalent definition of a TPS w.r.t. a collection of observables $\mathcal{A}_i \in L(\mathcal{H})$ as

1. The mutual commutativity of observables $[\mathcal{A}_i, \mathcal{A}_j] = 0 \quad \forall i \neq j$

[4]If one considers a hypergraph G, with vertices representing subsystems and edges representing interaction terms between subsystems, a Hamiltonian and correspondingly a TPS is k-local if there are at most k vertices joined by a single edge.

[5]In a Schmidt decomposition one can show that the entanglement entropy of either reduced density operator of a bipartitioned pure state is equal, $S(A) = S(B)$

2. The independence of observables $\mathcal{A}_i \cap \mathcal{A}_j = \mathbb{1}$
3. The generation of the algebra $\bigvee_i \mathcal{A}_i = L(\mathcal{H})$

4.2 Finite or infinite-dimensional Hilbert space?

The initial reason we chose to work in a finite-dimensional representation is from holographic principles; the entropy of a region is upper bounded by its surface and simultaneously the Hilbert space of that region is related to its entropy. There is still a problem of a finite dimensional representation that must be addressed. The classical argument for why we must abandon finite dimensional Hilbert space and embrace the infinite dimensional Hilbert space of QFT is due to the Canonical Commutation Relation (CCR)

$$[\hat{x}, \hat{p}] = i\mathbb{1}^n \,. \tag{4.2.1}$$

By then taking the trace of either side, and using the trace relation $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

$$\begin{aligned} \mathrm{tr}([\hat{x}, \hat{p}]) &= \mathrm{tr}(i\mathbb{1}^n) \\ \mathrm{tr}(\hat{x}\hat{p} - \hat{p}\hat{x}) &= i\mathrm{tr}(\mathbb{1}^n) \\ \mathrm{tr}(\hat{x}\hat{p}) - \mathrm{tr}(\hat{p}\hat{x}) &= in \\ \mathrm{tr}(\hat{x}\hat{p}) - \mathrm{tr}(\hat{x}\hat{p}) &= in \\ 0 &= in \,. \end{aligned} \tag{4.2.2}$$

Clearly we have reached an impasse unless we allow $\hat{x}$ and $\hat{p}$ to take infinite dimensional representations, such that $\mathrm{tr}(AB) \neq \mathrm{tr}(BA)$ i.e. unbounded operators. There is on the other hand a way of defining finite dimensional operators that do satisfy a CCR, if one defines the CCR in an exponential form. Herman Weyl proposed the CCR

$$e^{i\eta\hat{p}} e^{i\zeta\hat{x}} = e^{i\eta\zeta} e^{i\zeta\hat{x}} e^{i\eta\hat{p}} \,, \tag{4.2.3}$$

which admits finite dimensional representations of $\hat{x}$ and $\hat{p}$. Our choice of discretising momenta in Sec. 3.7 can then be justified, albeit with a less arbitrary construction, but suppose we would be forced to abandon a finite dimensional representation and rethink this entire framework in terms of an infinite dimensional representation? It turns out that mutual information is a well behaved, regularised quantity even in infinite dimensional representations, since for some non-overlapping sets A and B and a mutual information

$$I(A\!:\!B) = S(A) + S(B) - S(A \cup B) \,, \tag{4.2.4}$$

the divergencies in $S(A)$ and $S(B)$ can be made to be exactly cancelled by $S(A \cup B)$ [35].

4.3 Topological protection

The "strong claim" of ER = EPR was recently challenged by Verlinde [36], showing that black holes connected by an ER bridge need not be maximally entangled with mutual information exceeding classical limits. This is expressed by what is called Holographic principle for Black holes, which says that given the Bekenstein–Hawking relation can be interpreted as the amount of entanglement between two sides of an eternal black hole, the entropy associated with the spacetime region local to an ER bridge is that of a black hole in a mixed state which is topologically protected[37, 38]. By considering the time evolved TFD for a time t_α such that

$$e^{i(\alpha_n-\alpha_m)} = e^{-2i(E_n-E_m)t_\alpha} , \tag{4.3.1}$$

a state referred to as the Thermal Mixture Double (TMD) is constructed as

$$|TMD\rangle_\alpha = \sum_n e^{i(\alpha_n - \frac{\beta E_n}{2})} |n, n\rangle , \tag{4.3.2}$$

whose density matrix is the mixture

$$\rho_{TMD} = \sum_n e^{-i\beta E_n} |n, n\rangle\langle n, n| . \tag{4.3.3}$$

The entropy of this mixed state is that of a black hole, S_{BH} and so the mutual information between L and R is now classical, S_{BH}. In other words, the requirement of entanglement for the formation of an ER bridge is not necessary and it is therefore not clear that entangled states lead to the formations of ER bridges.

5 Summary

Any framework of emergent spacetime must faithfully reproduce the relative distances we can measure in the real world. When one first approaches emergent spacetime by means of mutual information between quantum systems, it may seem tempting to conclude that a maximally entangled quantum system like an EPR pair should then have a relative emergent distance of zero. Physics students around the world have for decades now performed labs involving EPR pairs being separated and measured that contradict this claim. We argue in the report that it is indeed the framing of the problem that is wrong and that a framework of emergent spacetime based on mutual information need not produce a zero distance between an EPR pair.

We do so by reviewing EPR pairs in their basic forms and computing their mutual information. We generalise this result for GHZ states and repeat this process by introducing basic models of the environment. We then introduce additional degrees of freedom, namely momentum, whose Hilbert space is much larger than the Hilbert space often considered for EPR pairs such as spin. Comparing the mutual information of entanglement in momentum space to that of spin space shows that in an emergent picture, entanglement in spin would have a much smaller effect on the total mutual information between an EPR pair due to the difference in sizes of the Hilbert spaces. On the other hand, we also show given a particular construction of the momentum states that even though the Hilbert space for momentum is much larger, the variation of the metric by modulation of spin entanglement is greater than the order of 10^{-3} making it feasible for experimental verification. We therefore indicate that experimental validation of a mutual information based emergent spacetime is theoretically feasible.

Introducing decoherence then shows how mutual information between two systems is lost to the environment. By introducing a thought experiment disentangling an EPR pair and entangling them with the environment, the Hamiltonian which determines the coupling to the environment could therefore leave the spin degrees of freedom highly entangled while decohere the momentum degrees of freedom, reducing the total mutual information between the systems while leaving the spin mutual information maximal. This in turn disentangles the environment from itself which in the emergent framework implies an increase in relative distance between two environments. We therefore also provide a qualitative mechanism for how in the macroscopic picture absolute distances arise i.e the distance between regions of spacetime.

To summarise, in a framework of emergent spacetime based on mutual information, a maximally entangled EPR pair would not yield a null distance in a metric because the mutual information between the particles would no longer be maximal once we include additional quantum degrees of freedom, in particular momentum. When we consider EPR pairs, we usually focus on the property of interest such as the spin or polarisation of particles and ignore the remaining degrees of freedom. Once included, the spin degrees of freedom contribute negligibly to the mutual information as shown in Sec. 3.6 since the Hilbert space of momentum is much larger than the Hilbert space of spin. As mutual information is proportional to $\ln \dim \mathcal{H}$ and the metric would be related to the negative logarithm of mutual information, the metric between the particles of a Bell pair would be effectively independent of the property which entangles the Bell pair, unless the Hilbert space of the entangled property is of the scale of the Hilbert space of momentum.

Appendix A

A.1 Tensor product structures

Definition *A Tensor Product Structure (TPS) is an equivalence class of isomorphism which endows Hilbert space with a factorisation as*

$$\mathcal{T} : \mathcal{H} \mapsto \bigotimes_i \mathcal{H}_i \,. \tag{A.1.1}$$

such that $\mathcal{T}_1 \sim \mathcal{T}_2$ if $\mathcal{T}_1\mathcal{T}_2^{-1}$ can be written as products of local unitaries $\bigotimes_i U_i$ up to permutation.

In this regard, operators acting only on their respective factors of Hilbert space introduces the notion of locality.

A.2 Pure states

Definition *Given a Hilbert space $\mathcal{H}$ the pure state $|\psi\rangle$ is any state that can be written as a linear combination of basis vectors that span $\mathcal{H}$. Given a TPS $\mathcal{T}$ satisfying $\mathcal{T} : \mathcal{H} \mapsto \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_K$, also known as a K-partite Hilbert space, a pure state can be written as*

$$|\psi\rangle = \sum_{i_1...i_K} a_{i_1...i_K} |i_1\rangle^{(1)} \otimes ... \otimes |i_K\rangle^{(K)} = \sum_{i_1...i_K} a_{i_1...i_K} |i_1, ..., i_K\rangle \tag{A.2.1}$$

As a shorthand, we define the tensor product state such that

$$|i_1\rangle \otimes ... \otimes |i_K\rangle \equiv |I\rangle \tag{A.2.2}$$

meaning that a pure state can be written as

$$|\psi\rangle = \sum_I a_I |I\rangle \tag{A.2.3}$$

If each state i_k is a qubit, we can use the binary mapping

$$|00...00\rangle \equiv |0\rangle \,, \quad |00...01\rangle \equiv |1\rangle \,, \quad |00...10\rangle \equiv |2\rangle \,, \quad ... \tag{A.2.4}$$

A.3 Density operators

Definition *Given a Hilbert space $\mathcal{H}$ the density operator ρ is an endomorphism $\rho : \mathcal{H} \mapsto \mathcal{H}$ mapping states in Hilbert space with a matrix representation acting on the Hilbert space with the following properties*:

(i) ρ is Hermitian

(ii) ρ is positive semi-definite

(iii) ρ has unity trace $tr(\rho) = 1$

A corollary of property (i) is that ρ can by a unitary transformation be decomposed to a real valued diagonal matrix, $D = U^\dagger \rho U$. *A density operator can represent either a **pure** state or a **mixed** state.*

Definition *A density operator representing a **pure** state $|\psi\rangle$ has the following properties:*

(i) $\rho = |\psi\rangle\langle\psi|$

(ii) $\rho^2 = \rho$

Notice that for the normalisation condition $\langle\psi|\psi\rangle = 1$, (ii) immediately follows from (i).

Definition *A density operator representing a* ***mixed*** *state cannot be written as the outer product of a single state vector, or projector, but rather refers to an ensemble of states* $|\psi_i\rangle$*. Instead of writing out this mixed state, we refer to a mixed state by the density opertor it is associated by*

$$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \ . \tag{A.3.1}$$

We can immediately see that we can construct mixed state density matrices as the superposition of projectors $P_i = |\psi_i\rangle\langle\psi_i|$

$$\rho = \sum_i p_i P_i \,. \tag{A.3.2}$$

A ***maximally mixed*** *state in a Hilbert space with* K *partitions is one that can be cast in the diagonal form*

$$\rho = \frac{1}{K}\mathbb{1}^K \,. \tag{A.3.3}$$

A pure state density operator can then be expressed as

$$\rho = |\psi\rangle\langle\psi| = \sum_{I,J} a_I a_J^* \, |I\rangle\langle J| = \sum_{I,J} a_{IJ} \, |I\rangle\langle J| \ . \tag{A.3.4}$$

Using Eq. (A.2.3) to form eigenstates

$$|\psi_n\rangle = \sum_I a_{I,n} \, |I\rangle \ , \tag{A.3.5}$$

we can from the Spectral theorem show that the mixed state density operators can be written as

$$\rho = \sum_n \lambda_n \, |\psi_n\rangle\langle\psi_n| = \sum_{n,I} \lambda_n a_{I,n} \, |I\rangle\langle I| = \sum_I b_I \, |I\rangle\langle I| \ . \tag{A.3.6}$$

Definition *A reduced density operator for some subset* $I_A \in I$*, where we partition the system as a multipartite system* $I = I_A \cup ... \cup I_Z$ *such that* $\rho = \rho_{A...Z}$*, is*

$$\rho_A = tr_{B...Z}(\rho_{A...Z}) = \sum_{L_B...L_Z} (\mathbb{1}_A \otimes \langle L|_B \otimes ... \otimes \langle L|_Z)\rho_{A...Z}(\mathbb{1}_A \otimes |L\rangle_B \otimes ... \otimes |L\rangle_Z) \,. \tag{A.3.7}$$

A.4 Entropy

Definition *Given a density matrix ρ, the entropy associated to this density matrix is given by the von Neumann entropy*

$$S = -tr(\rho \ln \rho)\,. \tag{A.4.1}$$

Since we can always diagonalise ρ, we can rewrite it in terms of the eigenvalues as

$$S = -\sum_i \lambda_i \ln \lambda_i\,. \tag{A.4.2}$$

We can then immediately see that if ρ represents the density matrix of a *pure* state

$$S = -1 \ln 1 = 0\,, \tag{A.4.3}$$

and if ρ represents the density matrix of a *maximally mixed* state of dimension k, the entropy is *maximal*

$$S = -\sum_i^k \frac{1}{k} \ln \frac{1}{k} = \frac{1}{k} \sum_i^k \ln k = \ln k\,. \tag{A.4.4}$$

The joint entropy S_{AB} for some bipartite density matrix ρ_{AB} is correspondingly

$$S_{AB} = -tr[\rho_{AB} \ln(\rho_{AB})]\,, \tag{A.4.5}$$

where if there is a decomposition such that $\rho_{AB} = \rho_A \otimes \rho_B$ then

$$\begin{aligned} S_{AB} &= -tr[(\rho_A \otimes \rho_B) \ln(\rho_A \otimes \rho_B)] \\ &= -tr[(\rho_A \otimes \rho_B)(\ln \rho_A \otimes \mathbb{1} + \mathbb{1} \otimes \ln \rho_B)] \\ &= -tr[\rho_A \ln \rho_A \otimes \rho_B + \rho_A \otimes \rho_B \ln \rho_B] \\ &= -tr[\rho_A \ln \rho_A \otimes \rho_B] - tr[\rho_A \otimes \rho_B \ln \rho_B] \\ &= -tr[\rho_A \ln \rho_A] tr[\rho_B] - tr[\rho_A] tr[\rho_B \ln \rho_B] \\ &= -tr[\rho_A \ln \rho_A] - tr[\rho_B \ln \rho_B] \\ &= S_A + S_B\,, \end{aligned} \tag{A.4.6}$$

and in general for

$$\rho = \bigotimes_i \rho_i \tag{A.4.7}$$

$$S(\rho) = \sum_i S(\rho_i)\,. \tag{A.4.8}$$

Theorem *The subaddativity of entropy states that*

$$S_{AB} \leq S_A + S_B \tag{A.4.9}$$

while the strong subadditivity of entropy states that

$$S_{ABC} + S_B \leq S_{AB} + S_{BC} \tag{A.4.10}$$

A.5 Mutual information

Definition *The mutual information between two subsystems A and B, or how much information about B one could ascertain by knowing A or vice versa, is given by*

$$I(A:B) = S_A + S_B - S_{AB} \tag{A.5.1}$$

with the following properties

(i) $I(A:B) \geq 0$, which can be shown through the subadditivity of entropy.

(ii) $I(A:B) \leq \ln(\dim(A)) + \ln(\dim(B))$, proof below

(iii) $I(X:YZ) \geq I(X:Y)$, which can be shown through the strong subadditivity of entropy.

(iv) $I(A:B) = I(B:A)$

(v) Is equivelant to the relative entropy $S(\rho_{AB}||\sigma_{AB})$ between the density matrix ρ_{AB} and the product of reduced density matrices $\sigma_{AB} = \rho_A \otimes \rho_B$.

The classical analog to (iv) is that it is a relative entropy between the joint probability distribution function $p(x, y)$ and an assumption of statistically independent product of distribution functions $p(x)p(y)$. In other words, it is the amount of uncertainty one has if one were to assume that the subsystems are completely independent.

Proof for (ii)
The from the subadditivity of entropy, joint entropy S_{AB} has the bounds

$$0 \leq S_{AB} \leq S_A + S_B\,, \tag{A.5.2}$$

so

$$0 \leq I(A:B) \leq S(A) + S(B)\,. \tag{A.5.3}$$

Since the individual entropies have the bounds

$$\begin{aligned} 0 &\leq S(A) \leq \ln(\dim(A)) \\ 0 &\leq S(B) \leq \ln(\dim(B))\,, \end{aligned} \tag{A.5.4}$$

the upper bound on mutual information is therefore

$$I(A:B) \leq \ln(\dim(A)) + \ln(\dim(B))\,. \tag{A.5.5}$$

A.6 Entanglement

Definition *If for a K-partite density operator*

$$\rho_{1..K} = \sum_{i_1...i_K} p_{i_1...i_K} \rho_1^{i_1} \otimes ... \otimes \rho_K^{i_K} \neq \sum_{i_1...i_K} \pi_{i_1} \rho_1^{i_1} \otimes ... \otimes \pi_{i_K} \rho_K^{i_K} \iff \forall i_n\, p_{i_1...i_K} \neq \pi_{i_1} ... \pi_{i_K}\,, \tag{A.6.1}$$

i.e. the coefficients are not separable, then $\rho_{1...K}$ represents an entangled state. Equivalently for a pure state, if a K-partite pure state $|\psi\rangle_{1...K}$ has the condition

$$|\psi\rangle_{1...K} = \sum_{i_1...i_K} a_{i_1...i_K} |i_1\rangle \otimes ... \otimes |i_K\rangle \neq \sum_{i_1...i_K} b_{i_1} |i_1\rangle \otimes ... \otimes b_{i_K} |i_K\rangle \iff \forall i_n\, a_{i_1...i_K} \neq b_{i_1} ... b_{i_K}\,, \tag{A.6.2}$$

then $|\psi\rangle_{1...K}$ is entangled.

A.7 Monogamy of entanglement

Definition *If A is entangled with B, then entangling C with A disentangles A and B.* Another way of wording this is that the mutual information between A and B decreases as C entangles with either A or B. In general, the definition can be expressed as [39]

$$\tau(\rho_{A_1(A_2....A_N)}) \geq \sum_{n=2}^{N} \tau(\rho_{A_1 A_n}) \,, \tag{A.7.1}$$

where τ is the square of the concurrence $\mathcal{C}$ for some bipartition ρ_{AB} [40]

$$\mathcal{C} \equiv \sqrt{2(1 - tr\rho_A^2)} \,. \tag{A.7.2}$$

A.8 Unitary transformations

Definition Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the operator U is an isometryd that maps $U : \mathcal{H}_1 \mapsto \mathcal{H}_2$ such that the inner product is preserved

$$\langle\psi|\psi\rangle = \langle U\psi|U\psi\rangle \ . \tag{A.8.1}$$

Some properties, including its matrix representation, are

(i) $U^{-1} = U^\dagger$.

(ii) $\det(U) = \pm 1$.

(iii) $U = e^{iH}$ where H is Hermitian.

Definition Given some TPS $\mathcal{T} : \mathcal{H} \mapsto \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_K$, a *local* unitary operator is one that can be cast in the form

$$U_{1..K} = U_1 \otimes ... \otimes U_K \ , \tag{A.8.2}$$

i.e. the operator acts individually on each subsystems. A consequence of this is that such an operator cannot couple or decouple any two subsystems i.e. *local operators preserve entanglement.* Conversely, any operator that does not preserve entanglement, i.e. entangles or disentangles any subsystems must be either a non-local unitary operator or a non-unitary operator (which could be local).

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