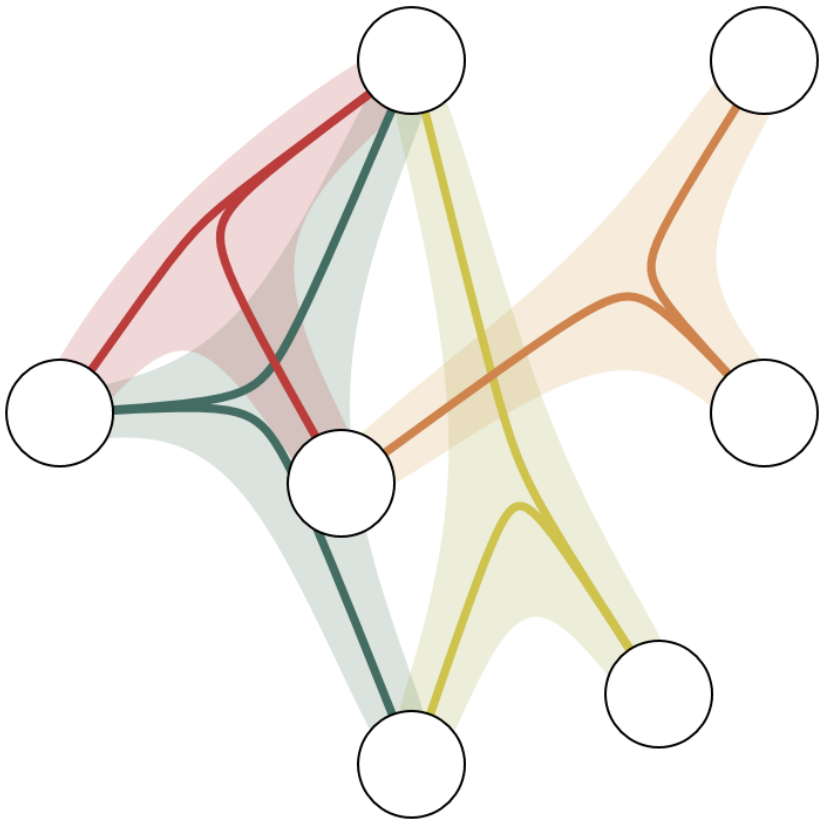
On quantum systems and the measurement problem

Nicolas Boulle



Supervisor: Dr. Guilherme Franzmann
Assistant supervisor: Dr. Sören Holst

Fysikum/Nordita
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Abstract

We focus on the Tensor Product Structure (TPS) of the Hilbert space and the fact that a choice in the TPS has an impact on the representation of the studied quantum system. We define the measurement problem in quantum mechanics and present some theories about quantum mechanics, each of them highlighting a different approach to quantum measurements. Then, a new approach to quantum measurement is presented by considering it as a change in the Tensor Product Structure of the Hilbert space associated with the description of a system. The system is made of a physical quantum system entangled with a measurement device. The description of the system changes to a new one where there is no entanglement anymore between the physical system and the measurement apparatus. The change in the TPS is performed using a global unitary transformation and more precisely by diagonalizing the density matrix of the system using unitary matrices. Four sets of matrices are obtained, each of them diagonalizing the density matrix in a different way for our toy model made of 2 qubits. Then, we want to recover Born's rule directly from the diagonalizing matrices by measuring the size of their sets using Haar measure. We have not been able to conclude this program, but we outline what is expected to happen such that standard probabilities can be recovered.

Keywords: Quantum physics, the measurement problem, quantum information, Hilbert spaces, physical systems, unitary, collapse, Euler angles parametrization, Copenhagen interpretation, many-world, measure.

Motivation

An important aspect in physics is to describe a physical system. Usually, a split is made between the system and its surrounding environment in such a way that the evolution of the system can be understood more easily and completely. The split depends on the best manner to describe the system and its surrounding environment.

In this thesis, we argue that the definition of a quantum system has an impact on the measurement procedure in quantum mechanics. A deeper understanding of a system might provide a new angle to understand the measurement problem.

We consider a simple model with a 2-qubit state and a measurement apparatus, each of them having a given factorization in the global Hilbert space corresponding to both the system and apparatus. We present a new approach to the measurement problem by considering that the measurement on the system induces a change of tensor product structure (TPS) which can change the Hilbert space factors originally associated with the system and the apparatus. Therefore, the description of the subsystem in the global Hilbert space itself is modified during the measurement process into a new description that is not in a superposition of states. The notion of system is not preserved, at least not in terms of its attribution as a given factor in Hilbert space. The corresponding factors of the system and the measurement apparatus in the Hilbert space are thus modified.

As a change of TPS is implemented by a unitary transformation, the measurement process is described in a unitary fashion. Nonetheless, we still seem to encounter a notion of collapse that would be related to the incorrect imposition that the factor in the Hilbert space of the system remains invariant throughout the measurement process. We hope to model this misattribution of the correct TPS in terms of information entropy in the future, such that the sum of entanglement and information entropy is conserved.

I want to thank my supervisor Guilherme, I learned a lot about many interesting things that I did not know before. I am happy to work on this challenging project and to explore new ideas. Also, I would like to thank the people working at Nordita, it was really pleasant to study in this nice work environment.

Contents

Introduction

Physics tries to describe and understand the behavior of physical systems. This can refer to mechanical motion, energy transfer, heat transfer, creation and annihilation of particles, the expansion of the universe, phase transition, or one of the many other examples.

The key to understand those quantities is to make use of physical systems. A physical system is a physical object or a collection of physical objects considered as a distinct entity which is studied using physics laws. The physical system is influenced by an environment that interacts against it, or does not (isolated system), through fundamental forces or interactions between the different systems. In classical mechanics, a system can be as simple as a ball, and a set of equations is going to describe the behavior of the system under the influence of, say, gravity. The ball is a physical system that exists in our world, one can touch the ball for example. The equations represent the physical description of the ball, which gives its degrees of freedom in a phase-space. Newtonian mechanics is then the main mathematical framework to describe the motion of a non-relativistic ball influenced by gravity. The physical system may be in different physical states: a solid object, a fluid or a gas, and the environment can impact it with temperature or pressure. Thermodynamics thus gives information about quantities such as heat capacity or entropy.

A system can also be a fundamental particle (particle physics) or the entire universe (cosmology). The cut between the system and the environment depends on a choice made by the observer to make equations simple and relevant to what the observer is willing to know. An observer who wants to know the trajectory of a ball is not going to consider the entire universe as a system, but maybe only the neighboring environment. The physical description of the ball can be seen as a collection of macrostates. There is a notion of coarse-graining: a macrostate is a state defined in such a way that its microscopic description is degenerate. Different microstates could result into the same macrostate, and the physics at the scale of the macrostate would remain the same. It is not necessary to study the properties of each particle making the ball to understand its motion in a mechanical classical framework.

In quantum mechanics, the classical notion of systems is changed to a notion of quantum systems. A system in quantum physics needs to be described with a vector state expressed in a given basis (or a density matrix for mixtures), a Hilbert space with a given factorization, and a Hamiltonian which generates the evolution of the system following Schrödinger's equation [1]. The evolution of closed quantum systems is unitary [2]. The vector state (or density matrix) is supposed to contain all the information about the system.

Quantum states belong to a Hilbert space: those are mathematical spaces that can be split and factorized into subspaces. The interest of factorizing Hilbert space is that the quantum system can be split in different ways depending on which observables are accessible [3]. Equations can appear easier using a certain factorization, and different criteria exist to choose and factorize the Hilbert space the best possible way [4, 5].

Quantum theory is used in many domains: quantum field theory for particles physics, quantum gravity, quantum information. Quantum information theory is interested in the transmission of information using quantum conveyors. The first time this idea appeared was in 1962 [6]. Quantum information theory makes use of the features from quantum physics: for example the fact that systems are described in a superposition of independent states or can be quantum correlated. Quantum information permits to send messages in a safer and more efficient way than with classical information theory.

Quantum systems can be measured. When measuring a quantum system, it seems that only one outcome is obtained for a given classical observer, who is then not observing superposition anymore. It is not well understood what is happening to the system when a measurement is performed and how the superposition disappears. The main interpretation of the measurement process is the Copenhagen interpretation [7] stating that a collapse happened: the wave function collapses to only one of the outcomes, the other states are not contained in the wave function anymore. Knowing how and whether a collapse

happened is the measurement problem [8]. There exists, besides the Copenhagen interpretation, different theories resolving the measurement problem such as the relative states or many world interpretation [9], or Bohmian mechanics [10, 11]. Those theories are facing difficulties: a collapse is not a linear evolution, the many-world interpretation suggests that every outcome occur when only one should, Bohmian mechanics states that the wave function does not contain all the information about the system [8].

In this thesis, we provide a background about quantum information theory (Chapter 1), the tensor product structure (TPS) of the Hilbert space (Chapter 2), and a summary about the measurement problem (Chapter 3). Then in Chapter 4 a new approach to the measurement problem is presented. This approach considers that the description of the quantum system in terms of Hilbert space factors is modified during a measurement process, as we consider a measurement as a change of TPS on the global Hilbert space. Therefore, the Hilbert space factorization is not the same before and after a measurement. We present a simple model: a two-qubit state. We use concepts from quantum information theory (density matrices, entanglement entropy) to present this idea and do the calculations. In Chapter 5, we speculate about a generalized second law of thermodynamics for quantum measurement.

Chapter 1

Quantum information

Quantum information is the study of the amount of information that can be conveyed by means of quantum systems. It is an interdisciplinary field which needs physicists, mathematicians, and computer scientists. Information theory has been mathematically developed mainly by Nyquist, Hartley and Shannon during the years 1920-1940. Shannon in particular had the idea of an information entropy giving the amount of information that can be stored into messages and sent to someone [12]. The fundamental unit of classical information is the bit. In quantum information, the fundamental unit is the quantum bit or qubit. The advantage of quantum information over classical information is that quantum phenomena can be used to send information, such as entanglement and superposition of states. A classical bit of information can only be in two different states: 0 *or* 1. However, a quantum bit can be in state 0, 1, or any superposition of these states. Quantum correlations are stronger than classical ones and allow an increase in the amount of information that can be carried. Protocols for quantum communications using quantum states as conveyor of the information have been developed [13, 14] and are a safer way to send information. However, it is more difficult to send quantum bits than classical ones, due to the high impact of the environment on the system that leads to decoherence. This can be avoided with quantum error correction codes such as the Shor's code [15].

In this chapter, we review some basic concepts of quantum information which will later be useful for the rest of the thesis.

1.1 Vectors in Hilbert spaces

When describing the physics of a system at the quantum level, we use the formalism of vectors and density matrices in Hilbert spaces. We consider in this thesis finite dimensional Hilbert spaces. A quantum state is in general a linear function carrying operators into complex numbers. A very convenient way of writing quantum mechanics is the bra-ket notation introduced by Paul Dirac in 1939 [16]. In quantum information, the Dirac formalism is generally used for its simplicity and clarity. Then, a state vector is to be written $|\cdot\rangle$, which can be written as a row vector. This is called a "ket". A line vector is written $\langle\cdot|$, called a "bra". Bra and ket belong to a Hilbert space [17], that we will, most of the time in this thesis, write $\mathcal{H}$.

Definition 1 *A Hilbert space is a linear complex vector space where a scalar product is defined. It can be infinite-dimensional. Vectors in the Hilbert space are denoted by* $|\phi\rangle, |\psi\rangle$ *etc... [18].*

We need the condition that, for each pair of elements (x, y) in the Hilbert space, a complex value $\langle x, y\rangle$ is associated to define the scalar product. The scalar product has the following properties:

1. The scalar product is conjugate symmetric: $\langle x, y\rangle = \langle y, x\rangle^*$.
2. It is linear in its first argument $\langle ax_1 + bx_2, y\rangle = a\langle x_1, y\rangle + b\langle x_2, y\rangle$.
3. $\langle x, x\rangle \geq 0$ and is 0 if and only if $x = 0$.

We can also refer to the bra as a dual vector. A dual vector is equivalent to the conjugate transpose of a vector: $\langle\phi| = \left(|\phi\rangle^*\right)^T = \left(|\phi\rangle^T\right)^*$.

A general state vector in a Hilbert space of dimension d is written

$$|\psi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle \ , \quad \langle\psi| = \sum_{i=1}^{d} \alpha_i^* \langle i| \ , \tag{1.1.1}$$

where α_i are complex numbers. Also, $\sum_{i=1}^{d} |\alpha_i|^2 = 1$ because the quantum state has to be normalized.

We mention a list of properties of vectors and dual vectors in Hilbert spaces:

1. $|c\phi + d\psi\rangle = c\,|\phi\rangle + d\,|\psi\rangle\,, \quad c \in \mathbb{C}$,
2. $\langle c\phi + d\phi| = \langle\phi|\, c^* + \langle\psi|\, d^*, \quad c \in \mathbb{C}$,
3. $||\phi|| = \sqrt{\langle\phi\,|\phi\rangle}$.

In quantum mechanics, we also need operators that act on the states. Indeed, a quantum state can evolve with time. Operators have to be unitary because of the linear evolution of closed quantum systems based on Schrödinger's equation. Also, a unitary evolution can be reversed in time as a unitary operator always has an inverse. For an operator $\hat{O}$, we need $\hat{O}\hat{O}^\dagger = \mathbb{1}$ where $\mathbb{1}$ is the identity. Operators are represented as $d \times d$ matrices where d is the dimension of the Hilbert space in the case of finite dimensional Hilbert spaces that we are considering in this thesis. Using the bra-ket formalism, an operator is written as:

$$\hat{O} = \sum_{i,j=1}^{d} O_{ij} |i\rangle \langle j| \ , \tag{1.1.2}$$

with O_{ij} a complex number with the condition $\sum_{j=1}^{d} O_{ij} O_{nj}^* = \delta_{in}$ for the operator to be unitary.

Therefore, the action of an operator over a state vector is

$$\hat{O}\,|\psi\rangle = \sum_{i,j=0}^{d-1} \sum_{i'=0}^{d-1} O_{ij} |i\rangle \langle j|\, \alpha_{i'} |i'\rangle = \sum_{i,j=0}^{d-1} O_{ij}\alpha_j |i\rangle \ , \tag{1.1.3}$$

which is a vector.

Quantum mechanics also describes composite systems. A composite system is a system containing subsystems, each of them being experimentally separately accessible. The Hilbert space for a composite system is generally a tensor product of Hilbert subspaces with each part of the system belonging to a different subspace. Therefore, a Hilbert space $\mathcal{H}^d$ of dimension d can be the tensor product factorization of many Hilbert spaces $\mathcal{H}^{d_i}$ of dimensions $d_i \leq d$ with the condition that $\prod_i d_i = d$. For a bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with vectors $|\phi\rangle \in \mathcal{H}_A$ and $|\theta\rangle \in \mathcal{H}_B$ the product vector is

$$|\psi\rangle = |\phi\rangle \otimes |\theta\rangle = |\phi\rangle\,|\theta\rangle = |\phi, \theta\rangle = |\phi\rangle_A\,|\theta\rangle_B \ , \tag{1.1.4}$$

and belongs to $\mathcal{H}_{AB}$. The product vector is linear in each of its argument. Vectors which cannot be written as product vectors are called entangled. For the same Hilbert space, different tensor product factorizations, or tensor product structures are generally accepted, as long as in each factorization the global Hilbert space has the same dimension. The global Hilbert space refers to the total space made by factorizing smaller-dimensional Hilbert spaces. It is then possible to decompose the Hilbert space in different ways depending on the needs to describe the quantum system (see Chapter 2).

Operators are defined such that they act on one or on many subspaces. A local operator only acts on individual factors Hilbert space:

$$(O_A \otimes O_B)\,|\phi\rangle_A\,|\theta\rangle_B = O_A\,|\phi\rangle_A \otimes O_B\,|\theta\rangle_B \ , \tag{1.1.5}$$

where each operator only acts on one of the subsystems.

An important kind of operator is the projection operator. It is used for projective measurements for example, which are measurements acting on the state vector. The initial state is projected onto a vector space using a projection operator P which has two properties: it is idempotent $P^2 = P$ and Hermitian $P = P^\dagger$. The first condition means that when a vector is projected onto a vector space, projecting it again onto the same vector space does not do anything. P is a positive operator, it has positive eigenvalues.

It can be written:

$$P = \sum_i p_i |i\rangle \langle i|\,, \quad p_i \geq 0\,, \tag{1.1.6}$$

with an orthonormal basis (ONB) $\{|i\rangle\}$ and $p_i \in \{0,1\}$. Therefore, the projection operator can also take the form:

$$P = \sum_{j\in I} |j\rangle \langle j| , \quad I \longleftrightarrow \text{subset of the ONB} . \tag{1.1.7}$$

The projection operator projects a vector onto the subspace I.

Let us consider a state $|\psi\rangle \in \mathcal{H}$:

$$|\psi\rangle = \sum_n \alpha_n |u_n^i\rangle . \tag{1.1.8}$$

The possible measured values of the observable A are the eigenvalues of A: $A|u_n^i\rangle = a_n |u_n^i\rangle$ where $i = 1 \dots g_n$. The probability $p(a_n)$ of getting a particular measured value a_n after a measurement is given using the projection operator P_n projecting onto the space of the eigenvectors corresponding to a_n:

$$p(a_n) = \langle\psi| P_n |\psi\rangle = \sum_{n',n''} \sum_{j=1}^{g_n} \alpha_{n'}^* \alpha_{n''} \langle u_{n'}^i| u_n^j\rangle \langle u_n^j| u_{n''}^i\rangle = |\alpha_n|^2 . \tag{1.1.9}$$

When a selective measurement of an observable on a state $|\psi\rangle$ gives an observed value a_n, the state following the measurement $|\psi'\rangle$ is given by:

$$|\psi\rangle \to |\psi_n'\rangle = P_n |\psi\rangle = \sum_j |u_n^j\rangle \langle u_n^j| \psi\rangle . \tag{1.1.10}$$

In this thesis we focus on finite dimensional Hilbert spaces and more particularly on $2-$dimensional qubit states.

1.2 Pure states and statistical mixtures

Composite systems can be of different kinds. A composite system represents a system made of different entities, such as different particles. It can also represent a system split into different parts: a physical system and a measurement apparatus, or an environment acting on a physical system. The different parts may or may not be correlated and the correlation may be of a quantum or a classical nature. We first focus on composite systems that do not exhibit quantum correlations. Those are written as product states. Consider a state $|\psi\rangle$ in $\mathcal{H}$ and two substates $|\phi\rangle$ and $|\theta\rangle$ in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Now consider that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

If the state $|\psi\rangle$ can be written

$$|\psi\rangle = |\phi\rangle_A \otimes |\theta\rangle_B , \tag{1.2.1}$$

then it is a product state, called separable state. It is written as the tensor product of the substates. The subsystems described by the substates are then non-correlated or only classically correlated.

A product state can be found in a pure state.

Definition 2 *A quantum system in a pure state is described by its state vector. It is a normalized vector.*

Pure states do not exhibit any classical correlations. They can be fully described using their state vector or their density operator ρ. We construct, from the pure state $|\psi\rangle$ living in $\mathcal{H}$, a matrix ρ called the density matrix

$$\rho \equiv |\psi\rangle \langle\psi| = (|\varphi\rangle_1 \langle\varphi|_1) \otimes (|\varphi\rangle_2 \langle\varphi|_2) . \tag{1.2.2}$$

If the global space is of dimension n, then the density matrix is a $n \times n$ matrix. It is also called a density operator.

Example: Let us consider a qubit state, that is a state in a two-dimensional Hilbert space. It can be in a superposition of two different states such as 0 and 1, or $\uparrow$ and $\downarrow$. The state is $|\varphi\rangle_1 = a|0\rangle + e^{i\alpha}\sqrt{1-|a|^2}|1\rangle, a \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. Now let us take a second state which is also a qubit: $|\varphi\rangle_2 = c|0\rangle + e^{i\gamma}\sqrt{1-|c|^2}|1\rangle, c \in \mathbb{C}$ and $\gamma \in \mathbb{R}$. A two-qubit state is obtained by taking the tensor product of those two substates:

$$\begin{aligned} |\psi\rangle &= (a|0\rangle + e^{i\alpha}\sqrt{1-|a|^2}|1\rangle) \otimes (c|0\rangle + e^{i\gamma}\sqrt{1-|c|^2}|1\rangle) \\ &= ac|00\rangle + ae^{i\gamma}\sqrt{1-|c|^2}|01\rangle + e^{i\alpha}\sqrt{1-|a|^2}c|10\rangle + e^{i(\alpha+\gamma)}\sqrt{1-|a|^2}\sqrt{1-|c|^2}|11\rangle . \end{aligned} \tag{1.2.3}$$

Calculating the norm of $|\Psi\rangle$, we conclude it is also a normalized state vector.

The density matrix of a pure state has the following properties.

1. it is positive: $\langle\phi|\rho|\phi\rangle \geq 0$. Therefore it is Hermitian: $\rho^\dagger = \rho$ and can thus be diagonalized using unitary matrices.
2. $\mathrm{Tr}[\rho] = 1$.
3. $\rho^2 = \rho$, (so $\mathrm{Tr}[\rho^2] = 1$).

Those can be checked explicitly for the density matrix of the state given in equation (1.2.3).

The expectation value of an observable A is determined using the density operator: $\langle A\rangle = \mathrm{Tr}[\rho A]$ [18]. The time evolution of the density operator is obtained using a unitary evolution:

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = U(t)|\psi(0)\rangle\langle\psi(0)|U^\dagger(t) = U(t)\rho(0)U^\dagger(t) . \tag{1.2.4}$$

This leads to the Schrödinger's equation for the density operator, or the von Neumann equation, depending on the system's Hamiltonian (Appendix A.1):

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)] . \tag{1.2.5}$$

Pure states are not the only kind of states in quantum mechanics. Statistical mixtures, also called mixed states or blends, are quantum states which cannot be written using vectors in Hilbert spaces. Consider a set of states $\{|\psi_i\rangle\}$ (i=1,...,N) which are obtained after preparation of similar quantum systems. Each $|\psi_i\rangle$ corresponds to a different preparation of the same system applied with a classical probability p_i. Those are all pure states. The statistical mixture is made of many states and we do not know the initial state of a statistical mixture perfectly well. To a statistical mixture is assigned a density operator:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| . \tag{1.2.6}$$

Statistical mixtures can be prepared in different manners. For example, when a non-selective measurement is made on a pure state, the result is a statistical mixture.

The properties of the density matrix for a statistical mixture are the following:

1. it is positive: $\langle\phi|\rho|\phi\rangle \geq 0$ and therefore it is Hermitian: $\rho^\dagger = \rho$ and can therefore be diagonalized using unitaries.
2. $\mathrm{Tr}[\rho] = 1$.
3. $\mathrm{Tr}[\rho^2] < 1$.

Therefore, the purity of a state can be determined by taking the trace of the density matrix squared since it does not have the same value for pure states and statistical mixtures. In the following section we are going to learn about a way to determine the purity of a state involving the trace of the squared density matrix: the von Neumann entropy.

1.3 The von Neumann entropy

In information theory, the Shannon entropy gives a measure of the information or uncertainty about the outcome of a random variable [12]. A random variable with a maximal Shannon entropy is a variable whose one has a maximal uncertainty about the outcome, and will convey a maximum amount of information to the receiver. Conversely, a message with entropy 0 is not surprising, and a minimum amount of information is conveyed since the receiver expected to receive such a message. The quantum version of Shannon entropy is the von Neumann entropy [19]. The von Neumann entropy gives information about the purity of a quantum state. Thus this entropy is a measure of the lack of information we have about a state. A pure state is showing all the needed information. However, a statistical mixture, as it is a convex sum of different states each of them with an associated classical probability, possesses a lack of information about which of the preparations made the state. There is more "surprise" for a receiver who gets a statistical mixture than for one who gets a pure state.

For a density matrix ρ, the von Neumann entropy is defined as:

$$S_{\mathrm{VN}} = -\mathrm{Tr}[\rho \ln \rho] \,. \tag{1.3.1}$$

The entropy does not change under unitary evolution (Appendix A.3):

$$S(\rho) = S\left(U\rho U^{\dagger}\right) \,, \tag{1.3.2}$$

and can also be written using the Schmidt decomposition (Appendix A.2):

$$S = -\sum_{n} p_n \ln(p_n) \,, \tag{1.3.3}$$

where p_n are the eigenvalues of the density operator.

The entropy is a concave function:

$$S\left(\sum_{i=1}^{k} \lambda_i \rho_i\right) \geq \sum_{i=1}^{k} \lambda_i S(\rho_i), \quad \sum_i \lambda_i = 1 \,. \tag{1.3.4}$$

Using equation (1.3.3), we conclude that a state with dimension d is maximally mixed for an homogeneous distribution $p_n = \frac{1}{d}$: $S_{VN} = \ln d$. In this thesis we are mainly working with qubit states, and the maximal von Neumann entropy is therefore $\ln 2$ for a single qubit. Indeed, you cannot write the 2-dimensional state represented by a density matrix

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \,, \tag{1.3.5}$$

as a state vector. The state is pure if its von Neumann entropy is 0, otherwise the state is in a statistical mixture.

1.4 Entanglement

The main feature that makes quantum mechanics different from classical mechanics is that two particles, or states, can be correlated in a way that the correlations are not of a classical nature. This is called entanglement. It is actually debated what makes quantum mechanics quantum, but the discussions generally converge to entanglement. It is a useful tool in quantum communication since an operation made on one of the entangled particles has an immediate impact on the other particles. Quantum entanglement corresponds to quantum correlations, also known as EPR correlations [20], between particles.

We say that two states are entangled when they cannot be written as a product state. For example, consider $|\phi\rangle$ in $\mathcal{H}_A$ and $|\theta\rangle$ in $\mathcal{H}_B$. The state $|\psi\rangle = |\phi\rangle \otimes |\theta\rangle$ in $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is not entangled because it is written as a product state. If such a way of writing $|\psi\rangle$ is not possible in a certain factorization of the Hilbert space, then there is entanglement between $|\phi\rangle$ and $|\theta\rangle$ in this factorization.

A state with density matrix ρ^{AB} that cannot be decomposed into the tensor product of the reduced density matrix

$$\rho^{AB} \neq \rho^{A} \otimes \rho^{B} \,, \tag{1.4.1}$$

is either classically correlated or quantum correlated (entangled).

A state whose reduced density matrices correspond to a pure density matrix is not entangled. Entangled states are pure states or mixtures with mixed reduced density matrices. A state classically correlated corresponds to a proper mixture of product states [18], that is a statistical mixture. Classically correlated states are created by LOCC (local operations classical correlations). Pure states that present correlations can only be entangled states, therefore, for a quantum correlated state:

$$\rho^{AB} \neq \sum_{r=1}^{m} p_r \rho^A \otimes \rho^B, \quad p_r \geq 0, \quad \sum_r p_r = 1\,. \tag{1.4.2}$$

Entanglement can also be seen using the Schmidt decomposition (Appendix A.2). Consider a pure state $|\Psi_{AB}\rangle$ in $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimensions a and b respectively. The density matrix of $|\psi\rangle$ is ρ^{AB} and the reduced density matrices are $\rho^A = \mathrm{Tr}_B[\rho^{AB}]$ and $\rho^B = \mathrm{Tr}_A[\rho^{AB}]$ where the trace is taken on one or the other subsystem.

Then, it is always possible to write using the Schmidt's decomposition:

$$|\psi\rangle = \sum_{n=1}^{k} \sqrt{p_n}\, |u_n^A\rangle\, |w_n^B\rangle\,, \text{ with } p_n > 0\,, \tag{1.4.3}$$

where $k \leq \min(a, b)$, $\{|u_n^A\rangle\}$ and $\{|w_n^B\rangle\}$ are the orthonormal basis of the eigenvectors of ρ^A and ρ^B respectively, and p_n are their eigenvalues (which are the same), called Schmidt coefficients. The number k of non-vanishing Schmidt coefficients is the Schmidt number. If the Schmidt number is equal to one, then the state $|\psi\rangle$ is separable, otherwise it is entangled. The Schmidt number can be used as a measure of the entanglement but there are other tools such as the entanglement entropy.

The entanglement entropy corresponds to the von Neumann entropy for the reduced density matrices. For pure states, the entanglement entropy is the same for all reduced density matrices, it does not depend on the local basis. The entanglement entropy of ρ^{AB} is obtained with the reduced density matrices and is

$$\begin{aligned} S = S^A = S^B &= -\mathrm{Tr}[\rho^A \ln \rho^A] = -\mathrm{Tr}[\rho^B \ln \rho^B] \\ &= -\sum_i \lambda_i \ln(\lambda_i)\,, \end{aligned} \tag{1.4.4}$$

where λ_i are the eigenvalues of the reduced density matrices. The maximal entanglement entropy for a two-qubit state is $\ln 2$ corresponding to a maximal entangled state, and 0 for a non-entangled state.

Example: For a two-qubit state of the form $\sqrt{p}\,|00\rangle + e^{i\phi}\sqrt{(1-p)}\,|11\rangle$, in $\mathcal{H}^2$ the entanglement entropy is given by $S(p) = -(p \log p + (1-p)\log(1-p))$. It has a maximum at $-(\log p + 1 - 1 - log(1-p)) = 0 \Leftrightarrow p = \frac{1}{2}$ which is $S(p)_{max} = \ln 2$. It can be seen on Figure 1.1 that the entanglement entropy is a concave function in this case.

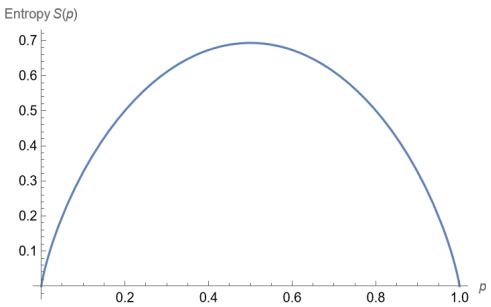


Figure 1.1: Entanglement entropy for a two-qubit state $\sqrt{p}\,|00\rangle + e^{i\phi}\sqrt{(1-p)}\,|11\rangle$. The maximum value of the entropy is $\ln 2$ obtained for $p = \frac{1}{2}$ corresponding to the inverse of the dimension of the Hilbert space.

Another useful definition is the linear entropy of entanglement. The presence of a logarithm in the expression of the entanglement entropy can make it difficult to find an analytical value in the case of

complex systems. Therefore, we can also use a linearized version of the entanglement entropy by making the approximation $\ln \rho = \rho - 1$

$$S_{lin} = 1 - \mathrm{Tr}[\rho_A^2] = 1 - \mathrm{Tr}[\rho_B^2]\,, \tag{1.4.5}$$

giving 0 for a separable state and $1 - 1/d$ for a maximally entangled state in d dimensions.

Let us now calculate the general formula of the entanglement entropy for a general qubit state:

$$|\psi\rangle = (a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle)\,, \tag{1.4.6}$$

the density matrix is

$$\rho = \begin{pmatrix} |a|^2 & ab^* & ac^* & ad^* \\ a^*b & |b|^2 & bc^* & bd^* \\ a^*c & b^*c & |c|^2 & cd^* \\ a^*d & b^*d & c^*d & |d|^2 \end{pmatrix}. \tag{1.4.7}$$

The reduced density matrices are

$$\rho_A = \begin{pmatrix} |a|^2 + |c|^2 & ab^* + cd^* \\ a^*b + c^*d & |b|^2 + |d|^2 \end{pmatrix} \quad \rho_B = \begin{pmatrix} |a|^2 + |b|^2 & ac^* + bd^* \\ a^*c + b^*d & |c|^2 + |d|^2 \end{pmatrix}. \tag{1.4.8}$$

Their eigenvalues will be the same. For ρ_A:

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 + 4(|(ab + cd)|^2 - (|a|^2 + |c|^2)(|b|^2 + |d|^2))}}{2}\,. \tag{1.4.9}$$

The solution for real coefficients is:

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4(ad - bc)^2}}{2}\,. \tag{1.4.10}$$

And the entropy

$$S = -\sum_{i=1}^{2} \lambda_i \ln \lambda_i\,. \tag{1.4.11}$$

Using real coefficients and the fact that $a^2 + b^2 + c^2 + d^2 = 1$, we parametrize the qubit with angles and write it as:

$$|\phi\rangle = \cos\alpha \sin\beta \cos\theta\,|00\rangle + \cos\alpha \sin\beta \sin\theta\,|01\rangle + \cos\alpha \cos\beta\,|10\rangle + \sin\alpha\,|11\rangle\,. \tag{1.4.12}$$

We see that in Figure 1.2 there is always an area of maximal entanglement and areas without entanglement. The entanglement entropy can also be visualized depending on the parameters used in $|\phi\rangle$ in the following animations made with Mathematica here. The Mathematica notebook is available in Appendix

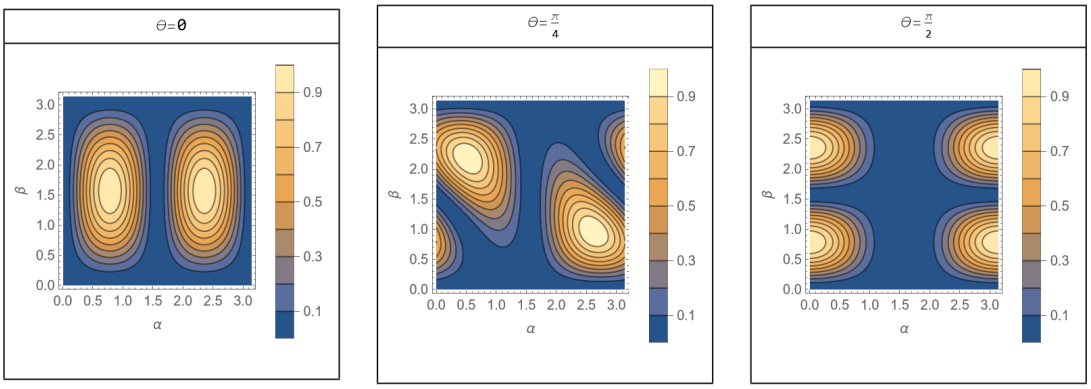


Figure 1.2: Entanglement entropy normalized for a qubit state $|\phi\rangle = \cos\alpha \sin\beta \cos\theta\,|00\rangle + \cos\alpha \sin\beta \sin\theta\,|01\rangle + \cos\alpha \cos\beta\,|10\rangle + \sin\alpha\,|11\rangle$. The axis of each plot are the angles α and β. The parameter θ varies between the different plots. Plotted using Mathematica.

D. The qubit states whose entanglement entropy is maximal are the Bell states. There exist 4 Bell states which are important for quantum information protocols:

$$|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} . \tag{1.4.13}$$

Bell states are highly correlated: a measurement on the first qubit in a Bell state determines the outcome of the second qubit for a measurement in the same basis.

1.5 Change of basis in the global Hilbert space

Entanglement is known to be basis-independent for local unitary basis-transformations. However, entanglement depends on how the different subsystems are separated from each other. A modification in the factorization of Hilbert subspaces has an impact on the entanglement of a state because this is equivalent to a change of basis in the global Hilbert space, and not a local change of basis. Therefore the subsystems are not separated from each other the same way as they were before. In a composite system, usually, each subsystem is written in a basis of its own Hilbert space, and the total Hilbert space is taken to be the tensor product of each subspaces. However, one can also decide to increase the dimension of some spaces, thus changing the properties of each subsystems. The global Hilbert space always has the same dimension, but the dimension of each of the subsystems can change. This can induce that entanglement is not visible anymore depending on the factorization.

Example: Let's take the qubit state $|\psi\rangle$ from equation (1.4.6) in the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimension 2. The coefficients are all complex numbers. The entropy of this system is given by equation (1.4.11).

Now, we make a change in the factorization of the Hilbert space but its dimension is kept to be the same. We make the changes $|00\rangle \to |a\rangle$, $|01\rangle \to |b\rangle$,$|10\rangle \to |c\rangle$ and $|11\rangle \to |d\rangle$.

The state is now written

$$|\phi'\rangle = a\,|a\rangle + b\,|b\rangle + c\,|c\rangle + d\,|d\rangle \ . \tag{1.5.1}$$

We now see the system as a 4-levels system. The degrees of freedom of the system are packed in only one system, there is no notion of subsystems. Here, it is easy to notice that there is no entanglement anymore, even if the initial state was entangled (by taking, for example, a Bell state with $a = d = \frac{1}{\sqrt{2}}$ and $b = c = 0$). So in this particular factorization of the global Hilbert space, the system will never be entangled because there is only one system and no subsystems. A system that cannot be entangled is not useful to transmit quantum information.

Chapter 2

Tensor product structure

Let us consider the following state:

$$|s\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\,. \tag{2.0.1}$$

If we do not describe the Hilbert space in which the state vector is expressed, it is impossible to describe the system. We may think that it is a 2-qubit state in $\mathcal{H}^4 = \mathcal{H}^2 \otimes \mathcal{H}^2$, but we do not actually know if we are here working with qubits. The first Hilbert space could be of dimension 4, and the second of dimension 2, giving a total space of dimension 8: $\mathcal{H}^8 = \mathcal{H}^4 \otimes \mathcal{H}^2$, and the state could be written:

$$|s_a\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) + 0(|00\rangle + |11\rangle + |20\rangle + |21\rangle + |30\rangle + |31\rangle)\,. \tag{2.0.2}$$

It is therefore important to precise the Hilbert space and particularly its dimension, otherwise it can induce errors in the interpretation of the state.

The Hilbert space can also be factorized in different manners. The factorization of the Hilbert space in subspaces is needed to provide the description of the system. For example, we could interpret a Hamiltonian as the Hamiltonian of a particle in two dimensions when it would actually be the Hamiltonian of two particles in one dimension.

A Hilbert space of dimension 8 can be split in different ways: for example $\mathcal{H}^8 = \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2$. Writing the state $|s_a\rangle$ in this factorization could give:

$$|s_b\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |010\rangle) + 0(|000\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)\,. \tag{2.0.3}$$

Writing the same state in a Hilbert space which has no split gives:

$$|s_c\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) + 0(|0\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle)\,. \tag{2.0.4}$$

In each case, the elements of the state belong to an 8-dimensional global Hilbert space. In the case of equation (2.0.4), there can not be entanglement as there is only one Hilbert space and no subsystems. But in equations (2.0.2) and (2.0.3), entanglement can be seen and the states are actually maximally entangled. Therefore, entanglement entropy disappears when we change the way the Hilbert space is constructed.

The decomposition of the Hilbert space is important in the description of the system in order to know the dimension of each constituent of the system, as well as the potential interactions between the different parts of the system. A different factorization brings a different description of the same system and different degrees of freedom. The understanding of a system depends on how subsystems are defined from the total system. As we will see, the factorization depends on the observer, and on the information they want to know about the quantum system. In classical mechanics, the observer has to make a split between the physical system and its environment in order to make observations and analysis in a relevant way. We are going to see that in quantum mechanics the observer can split the total Hilbert space in different ways, and have different pictures of the same system.

2.1 Definition

We have seen that it is important to give the dimension and the factorization of the Hilbert space associated with a quantum system to understand it. We refer to different factorization as different tensor product structures (TPS).

Different TPS's are allowed for the same Hilbert space, as mentioned earlier, and depending on the partition of the space, entanglement exists or vanishes. It would not be more efficient to send a message using quantum states in a TPS where entanglement is not visible, compared to a classical way. Furthermore, each factorization gives a new description of the quantum system with new accessible observables and interactions between the parts of the system. Then, how can one choose a preferable TPS? Which partition of the Hilbert space gives the most understandable description for the studied system? There exist different criteria to choose the preferable TPS, and we review some of them in this chapter.

There are different definitions for the TPS. The first is given in [4] and permits to obtain a TPS on a Hilbert space that is not written as an explicit tensor product:

Definition 3 (TPS) *A TPS $\mathcal{T}$ of a Hilbert space $\mathcal{H}$ is an equivalence class of isomorphism T: $\mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots$ where $T_1 \sim T_2$ whenever $T_1 T_2^{-1}$ may be written as a product of local unitaries $U_1 \otimes U_2 \otimes \dots$ and permutations of subsystems.*

T is an isomorphism that makes us able to talk about locality in a subsystem. When $T_1 T_2^{-1} = U_1 \otimes U_2 \otimes \dots$ up to permutations of subsystems, the two TPS's have the same structure, and show the same way of factorizing the Hilbert space. That corresponds to the situation when the subsystems of two different TPS's are related by unitary transformations and permutations. Reorganizing the subsystems does not mean that the TPS has changed : $\mathcal{H}_A \otimes \mathcal{H}_B$ has a TPS similar to $\mathcal{H}_B \otimes \mathcal{H}_A$. Only the internal structure of the Hilbert space is changing, but the space is the same in its globality. An analogy is presented in [4]: the change of TPS can be viewed as a change of coordinates. A set of coordinates can be more useful when considering a certain system, than another.

A TPS is relative to the accessible observables and their degrees of freedom, in other words, to the experiments that can be done on the system. Then, the TPS is constructed directly from the algebraic structure of the observables [3]. A TPS is also defined as [4, 3]:

Definition 4 (TPS) *A TPS on a Hilbert space $\mathcal{H}$ is a collection of subalgebras $\{\mathcal{A}_i\}$, $\mathcal{A}_i \in L(\mathcal{H})$ satisfying the three axioms*

1. *$[\mathcal{A}_i, \mathcal{A}_j] = 0$ for $i \neq j$,*
2. *$\mathcal{A}_i \cap \mathcal{A}_j = \mathbb{1}$ ($\mathcal{A}_i$ are independent),*
3. *The $\mathcal{A}_i$ generate the whole algebra (the whole accessible physical measurements).*

The subalgebras are the observables and interactions accessible by the system, for example spin measurement. They act locally on subspace i and have the form

$$\mathcal{A}_i = \mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes O_i \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}\,, \tag{2.1.1}$$

where O_i represents an operator acting on one subsystem. The observable is acting on subsystem i through a unitary operator, and has no effect on the other subsystems. The first condition ensures that the observables are separable: one can perform a local operation on one of the subsystems without affecting another subsystem. The second condition means that the observables are independent with respect to each other. The last condition confirms that the collection of subalgebras corresponds to all the accessible observables. Then the TPS is the collection of subalgebras. Using the set of accessible observables, one can determine the structure of the Hilbert space:

Proposition 1 *A set of subalgebras $\mathcal{A}_i$ satisfying Axioms 1.-3. induces a TPS $\mathcal{C} = \otimes_{i=1}^{n} \mathcal{H}_i$. We call such a multi-partition an induced TPS [3].*

Therefore, the number of elements in the collection of subalgebras determines the number of Hilbert subspaces, and the way to split the Hilbert space. The chosen factorization is induced by a set of accessible observables. There can be many different sets and therefore many different factorizations with different degrees of freedom that can be chosen for the same quantum system. An observer willing to

know an observable of a quantum system has to split the Hilbert space in a TPS where this observable is relevant.

Depending on the TPS, we will obtain different properties for the studied system. Entanglement is relative to a set of observables and is not an absolute notion that always exists in the system.

2.2 Choice of TPS

We have seen that for a total Hilbert space of dimension 8 , there can be many different choices of factorization. This is the case for all Hilbert spaces with a dimension which is not a prime number. The chosen TPS is the one that will bring the most information to the observer, the one describing the system in a way that equations are easier to understand, and carry a meaningful description of the physical system. There exist different criteria to choose the best TPS. Two criteria will be presented here: a dynamical criterion [4], and a kinematical [5]. Both argue on the best way to factorize the Hilbert space depending on the needs and operations that have to be carried out over the system. The first criteria invokes the idea of locality of the system's Hamiltonian, and the second the idea of quasi-classicality of the system's evolution. We briefly present them in this section.

2.2.1 Dynamical criterion

The concepts in this section come from [4].

The first criterion presented is to choose a TPS $\mathcal{T}$ where the Hamiltonian looks local. By local, it is here meant that only a few collection of subsystems are interacting. The local degrees of freedom of the Hamiltonian can almost always be obtained by reading its energy spectrum, and we can then write the Hamiltonian in its local form if the only information given is its spectrum.

In this section, a TPS where the Hamiltonian is acting on, for example, 2 subsystems over a total of 8, carries a simpler, or more meaningful description of the system than those where it is acting on all of the subsystems, or a big amount. A reason to choose locality as a criterion is advanced by the authors in [4]: the classical world around us looks local in a spacetime sense[1]. Then, it is more natural to choose a factorization that will make the Hamiltonian look local, and consider a system evolving locally.

A Hamiltonian acting on n qudits is written as :

$$H = a_0\mathbb{1} + \sum_{i=1}^{n}\sum_{\alpha=1}^{d^2-1} a_\alpha^i O_i^\alpha + \sum_{i<j}\sum_{\alpha,\beta=1}^{d^2-1} a_{\alpha\beta}^{i,j} O_i^\alpha O_j^\beta + \sum_{i<j<k}\sum_{\alpha,\beta,\gamma=1}^{d^2-1} a_{\alpha\beta\gamma}^{ijk} O_i^\alpha O_j^\beta O_k^\gamma + \dots , \tag{2.2.1}$$

where O_i^α are operators that form an orthogonal basis. The basis contains the single-qudit operators acting on qudit i, each operator only acts on one subsystem. The second term $O_i^\alpha O_j^\beta$ acts on 2 different qudits. In this section and in [4], a local Hamiltonian is a Hamiltonian acting on a small amount of subsystems.

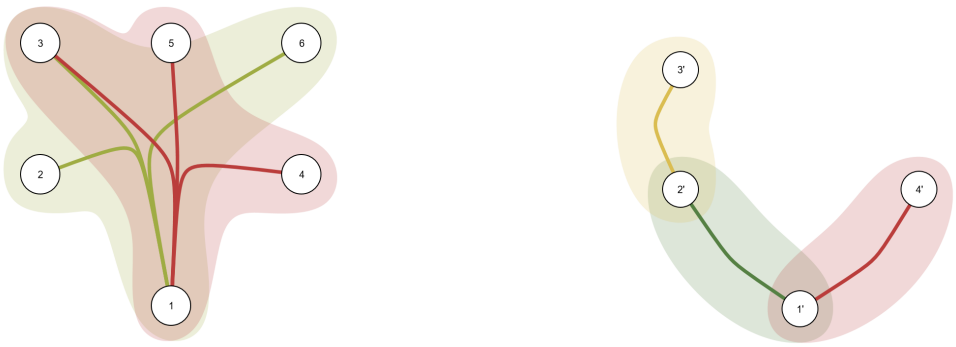
A local Hamiltonian is not the same as a local operator. A local operator acts only on a single subsystem or on a collection of subsystem. A local Hamiltonian is a sum of local operators, the locality of the Hamiltonian is only related with the amount of subsystems interacting.

Locality can be viewed using a hypergraph. We represent the subsystems by a set of vertices V and the Hamiltonian by a collections of edges $E = \{E_i\}$. In a normal graph, an edge is written $E_i = \{v, v'\}, \quad v, v' \in V$. In a hypergraph, the edges can connect more than two vertices. Then, the number of vertices connected through the edges represent the subsystems the Hamiltonian is acting on. A Hamiltonian acting on only one subsystem is called 1-local. A Hamiltonian acting on k subsystems is called k-local. Here, a local Hamiltonian would correspond to a Hamiltonian with a small k compared to the number of subsystems.

For a given Hamiltonian H, a choice of TPS $\mathcal{T}$ produces a Hamiltonian THT^{-1} on $\bigotimes_i \mathcal{H}_i$. The Hamiltonians are the same, in the sense that they have the same spectrum, however they are expressed in different TPS's. Therefore, we have to talk about a Hamiltonian with respect to a given TPS. We say that two TPS's $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent to a given Hamiltonian H if $T_1 H T_1^{-1}$ and $T_2 H T_2^{-1}$ are the same up to conjugation by local unitaries, permutation of the subsystems and transposition.

[1] The world around us looks local in a spacetime sense, not in a quantum mechanical sense. For instance, take gravity: everything is always interacting gravitationally, then we would imagine that quantum mechanically we would write a Hamiltonian where all the systems are interacting, being very non-local. Nonetheless, there are ideas to connect the locality from quantum mechanics and the locality in spacetime [21]

Example: Consider a d-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5 \otimes \mathcal{H}_6$. The Hamiltonian $\hat{H} = \hat{O}_1 \otimes \hat{O}_2 \otimes \hat{O}_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \hat{O}_6 + \hat{O}_1 \otimes \mathbb{1} \otimes \hat{O}_3 \otimes \hat{O}_4 \otimes \hat{O}_5 \otimes \mathbb{1}$ is 4−local as it is acting on a maximum of 4 subsystems in each of its terms. However, expressing the Hamiltonian in a different TPS $\mathcal{H} = \mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \mathcal{H}'_3 \otimes \mathcal{H}'_4$ could give $\hat{H} = \hat{O}'_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \hat{O}'_4 + \mathbb{1} \otimes \hat{O}'_2 \otimes \hat{O}'_3 \otimes \mathbb{1} + \hat{O}'_1 \otimes \hat{O}'_2 \otimes \mathbb{1} \otimes \mathbb{1}$ which is a 2-local Hamiltonian. According to the dynamical criterion, this TPS offers a better description of the system, because the Hamiltonian looks more local: it acts on half of the subsystems. It can be viewed in Figure 2.1.



(a) (b)

Figure 2.1: Hypergraph for the 2-local and 4-local Hamiltonians $\hat{H} = \hat{O}_1 \otimes \hat{O}_2 \otimes \hat{O}_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \hat{O}_6 + \hat{O}_1 \otimes \mathbb{1} \otimes \hat{O}_3 \otimes \hat{O}_4 \otimes \hat{O}_5 \otimes \mathbb{1}$ acting on a state in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5 \otimes \mathcal{H}_6$ and $\hat{H} = \hat{O}'_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \hat{O}'_4 + \mathbb{1} \otimes \hat{O}'_2 \otimes \hat{O}'_3 \otimes \mathbb{1} + \hat{O}'_1 \otimes \hat{O}'_2 \otimes \mathbb{1} \otimes \mathbb{1}$ acting on a state in $\mathcal{H} = \mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \mathcal{H}'_3 \otimes \mathcal{H}'_4$. Each vertex represents a subsystem. A vertex named i represents a subsystem in $\mathcal{H}'_i$. The edges represent the interaction Hamiltonian.

Duality

The Hilbert space can be factorized using different TPS's. But different TPS's for the same Hilbert space can be equivalent with respect to the system's Hamiltonian.

It can appear that a Hamiltonian expressed in two non-equivalent TPS's have the same k-locality. Two non-equivalent TPS's with respect to a Hamiltonian, where the Hamiltonians are k-local are called dual. Equivalently, two Hamiltonians in the same TPS are dual if they are k-local, have the same spectrum and are not related by local unitaries transformation, permutations of subsystems and transposition. In Figure 2.1, the two TPS's are different, and the Hamiltonian in Figure 2.1a is less local than the other TPS from Figure 2.1b, as it acts on a relatively big amount of subsystems compared to the other. Therefore, the two TPS's are not dual with respect to this Hamiltonian (if those two TPS's are non-equivalent). Generic local Hamiltonians do not have duals: in the same TPS, it is not possible to find two non-equivalent local Hamiltonians with the same spectrum.

An example of duality: For a given Hamiltonian, two descriptions are dual if both TPS's are non-equivalent and the Hamiltonians have the same k-locality. For example, take the Hamiltonian for a harmonic oscillator in a 1-dimensionnal quantum harmonic chain made of atoms:

$$\sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 \sum_{\{i,j\}} (x_i - x_j)^2 \,, \tag{2.2.2}$$

where the mass m is assumed to be the same for all atoms. p_i and x_i are the momentum and position for the i-th atom. If we make a change of variable using a discrete Fourier transform to

$$Q_k = \frac{1}{\sqrt{N}} \sum_l e^{ikal} x_l \,, \tag{2.2.3}$$

and

$$\Pi_k = \frac{1}{\sqrt{N}} \sum_l e^{-ikal} p_l \,, \tag{2.2.4}$$

where Q_k are the normal coordinates and Π_k the conjugate momenta, the Hamiltonian is then written

$$H = \frac{1}{2m}\sum_k (\Pi_k \Pi_{-k} + m^2\omega_k^2 Q_k Q_{-k})\,. \tag{2.2.5}$$

Those two descriptions of the Hamiltonian are dual. Indeed, the change in the set of coordinates induces a change in the TPS and the new coordinates are non-locally related to the old ones but both descriptions of the Hamiltonian have the same k-locality.

An example of non-duality: In this example, we consider a Hamiltonian H in a 4-dimensional Hilbert space $\mathcal{H}$ with TPS $\mathcal{T}$. We make the Hamiltonian unitarily vary and keep the same TPS. We shall see that the obtained Hamiltonians are not dual.

We take a simple Hamiltonian:

$$H^{AM} = g\sigma_z^A \otimes \sigma_y^M\,, \tag{2.2.6}$$

with σ_i are the Pauli matrices. The system is in a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_M$ where $\mathcal{H}_A$ and $\mathcal{H}_M$ are 2-dimensional Hilbert spaces.

This TPS is induced by the accessible observables. From the Hamiltonian, we notice that there are interactions between the z and y components of the qubits.

The Hamiltonian, written in matrix form, is

$$H^{AM} = gi\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{2.2.7}$$

With eigenvalues for H^{AM}: $\{-g, g\}$ with degeneracy 2. The Hamiltonian is Hermitian and can therefore be diagonalized using a global unitary transformation:

$$\begin{aligned} H_1 &= g\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g\sigma_z \otimes \sigma_z\,, \\ H_2 &= g\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g\mathbb{1} \otimes \sigma_z\,, \\ H_3 &= g\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g\sigma_z \otimes \mathbb{1}\,, \\ H_4 &= g\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.2.8}$$

The diagonalization of the Hamiltonian into H_4 provides a 4-level system without any factorization.

We summarize how the different Hamiltonians for each TPS are related in Table 2.1 We therefore conclude that the Hamiltonians are not dual with respect to this TPS.

It is then possible, after diagonalization, to write the Hamiltonian in different forms that are not dual. Here, we can write them either as the interaction of two σ_z or without interaction between the spins qubits at all.

It is equivalent to consider a Hilbert space $\mathcal{H}$ with a fixed Hamiltonian H and varying choice of TPS, or a Hilbert space $\mathcal{H}$ with a fixed TPS $\mathcal{T}$ and varying choices of Hamiltonian. We can therefore consider that the 4 obtained Hamiltonians are the same, but expressed in different TPS's. Thus, by diagonalizing the Hamiltonian, we either: find a new Hamiltonian in the same TPS *or* find the same Hamiltonian but expressed in a different TPS.

The graphs for the Hamiltonians H_2 and H_3 correspond to two vertices without hyper-edges linking them. The graphs associated to the other TPS's represent two vertices with one hyper-edge linking them.

The TPS's where the Hamiltonian is expressed as H_2 and H_3 are the same by permutations. However, the other TPS's are different: they cannot be related by local unitaries and permutations of subsystems.

In [4], it is argued both analytically and numerically that typically a generic Hamiltonian cannot be written in any $k-local$ TPS. Said differently, in a given TPS, there are no any $k-local$ Hamiltonians with the same spectrum. The TPS where the Hamiltonian is k-local is generally unique.

Hamiltonian	Same k-locality	Related by local unitaries	Related by permutations	Related by transpositions	Dual?
H^{AM}/H_1	No	No	No	No	No
H^{AM}/H_2	No	No	No	No	No
H^{AM}/H_3	No	No	No	No	No
H^{AM}/H_4	No	No	No	No	No
H_1/H_2	No	No	No	No	No
H_1/H_3	No	No	Yes	No	No
H_1/H_4	No	No	No	No	No
H_2/H_3	Yes	No	Yes	No	No
H_2/H_4	No	No	No	No	No
H_3/H_4	No	No	No	No	No

Table 2.1: Comparison of the Hamiltonians in TPS $\mathcal{T}$. The Hamiltonian $H^{AM} = g\sigma_z^A \otimes \sigma_y^M$ and H_1, H_2, H_3 and H_4 are diagonalized versions of H^{AM}. Those Hamiltonians are non-dual in TPS $\mathcal{T}$.

We can generalize that to a finite 2^n-dimensional Hilbert space $\mathcal{H}$. We consider the Hamiltonian:

$$H = \bigotimes_{i=1}^{n} \sigma_i \,, \tag{2.2.9}$$

with $\sigma_i \in \{\sigma_x, \sigma_y, \sigma_z\}$. The spectrum of such a Hamiltonian corresponds to the set $\{-1, 1\}$ with degeneracy 2^{n-1}. It is always possible to write it as the tensor product between m matrices σ_z and p Identities with $m + p = n$ and $m \geq 1$. Therefore, a Hamiltonian made of Pauli matrices always has a TPS where it is 1-local.

2.2.2 Kinematical criterion

The concepts in this section come from [5].

The other criterion is a kinematical one. The choice of the Hilbert space factorization is determined by the one in which the system evolves quasi-classically under the influence of the environment. Here, the Hilbert space is divided in two parts: a physical quantum system and an environment monitoring the evolution of the physical system considered.

Consider a system made of a quantum system S, a quantum measurement apparatus A in some basis, and an environment E monitoring the system. The environment has an impact on the measurement apparatus, and the interaction Hamiltonian of the system is of the form :

$$\hat{H} = \hat{H}_{\mathrm{SA}} + \hat{H}_{\mathrm{EA}} \,. \tag{2.2.10}$$

It is considered that the environment has no impact on the system, only on the apparatus, otherwise the system would continue to evolve even after the measurement is done. A *pointer basis*, is the basis of eigenvectors corresponding to an observable $\hat{O}$, such that

$$[\hat{O}, \hat{H}_{\mathrm{EA}}] = 0 \,, \tag{2.2.11}$$

the observable commutes with the interaction Hamiltonian between the environment and the apparatus. Therefore, this observable is not perturbed by the environment. The pointer basis defines which observable can be recorded by the apparatus under evolution monitored by the environment [22, 23].

A quasi-classical evolution of the state is achieved when entanglement between the environment and pointer states of the system does not increase too much during their interaction. Furthermore, their trajectories have to remain near classical trajectories.

Consider a d-dimensional Hilbert space $\mathcal{H}$ without any particular factorization. The only other given information is the Hamiltonian H with its spectrum and an initial state $|\psi(0)\rangle$. The structure of the Hilbert space is recovered from this information and the choice of factorization comes from the imposition that the initial state evolves quasi-clasically. Quasi-classicallity is defined using two criteria: the robustness of the state against entanglement with the environment, which is quantified using the entanglement linear entropy, and the predictability of dynamics, imposing that the state does not spread too much [24].

We present the first criterion. We consider a split of the Hilbert space into a system and the environment:

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E \,. \tag{2.2.12}$$

At initial time, the system is not entangled with the environment and we wish to get a factorization of the Hilbert space so that the system stays as little entangled as possible under influence of a Hamiltonian $\hat{H}$. The interaction term of the Hamiltonian, representing the interaction between the environment and the system, is much stronger than the local free terms. Therefore, the evolution of the system is mostly monitored by the effect of the environment. This is called the Quantum Measurement Limit [25].

A good way to quantify entanglement, as seen in Chapter 1, is to make use of the von Neumann entropy for the system under the influence of the environment (entanglement entropy). The problem with the entanglement entropy is that it contains a logarithm inside of it, which makes it difficult to obtain an analytical expression. Instead, the linear entanglement entropy is used

$$S_{lin} = 1 - Tr[\rho^2] \,. \tag{2.2.13}$$

Consider an arbitrary factorization of the Hilbert space $\mathcal{H} = (\mathcal{H}_A \otimes \mathcal{H}_B)_{\{\theta\}}$ where θ is "a factorization", and a product state $\rho(0) = |\psi_A(0)\rangle \langle\psi_A(0)| \otimes |\psi_B(0)\rangle \langle\psi_B(0)|$. Using a unitary evolution and an interaction Hamiltonian of the form $H_{int} = \lambda_{\{\theta\}} (A \otimes B)$, the entropy is[2], up to $\mathcal{O}(t^2)$:

$$S_{lin}(\rho_A(t)) = 2\lambda_{\{\theta\}}^2 t^2 \left(\langle A^2\rangle_0 - \langle A\rangle_0^2\right)\left(\langle B^2\rangle_0 - \langle B\rangle_0^2\right) . \tag{2.2.14}$$

Therefore, the entropy depends on the interaction strength of the Hamiltonian, but also on the spread of the initial state with respect to the terms in the Hamiltonian. The best way to factorize the Hilbert space using this result would be to take a factorization where the strength of the interaction Hamiltonian is low but also where the initial state does not spread too much.

In conclusion, different criteria exist to choose the more relevant way to factorize the Hilbert space, depending on the needs of the observer. The quasi-classical factorization is useful in some definite cases. However, the locality criterion takes into account that the observer is local. We see in Chapter 4 that the measurement problem could be related to the problem of finding the best TPS.

2.3 Time evolution

Let us now see what is happening when we change the TPS for a state and a Hamiltonian. We first consider a Hamiltonian H and a state $|\psi(0)\rangle$ in a 4-dimensional composite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimension 2. We perform a unitary evolution of the system.

Let us take the Bell's state: $|\psi(0)\rangle = |\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and the Hamiltonian from equation (2.2.6) $H = g\sigma_z \otimes \sigma_y$.

The time evolution of the system is given by

$$|\psi(t)\rangle = e^{-itH/\hbar}|\psi(0)\rangle = \begin{pmatrix} \cos\left(\frac{gt}{\hbar}\right) & -\sin\left(\frac{gt}{\hbar}\right) & 0 & 0 \\ \sin\left(\frac{gt}{\hbar}\right) & \cos\left(\frac{gt}{\hbar}\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{gt}{\hbar}\right) & \sin\left(\frac{gt}{\hbar}\right) \\ 0 & 0 & -\sin\left(\frac{gt}{\hbar}\right) & \cos\left(\frac{gt}{\hbar}\right) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\cos\left(\frac{gt}{\hbar}\right)}{\sqrt{2}} \\ \frac{\sin\left(\frac{gt}{\hbar}\right)}{\sqrt{2}} \\ \frac{\sin\left(\frac{gt}{\hbar}\right)}{\sqrt{2}} \\ \frac{\cos\left(\frac{gt}{\hbar}\right)}{\sqrt{2}} \end{pmatrix} , \tag{2.3.1}$$

[2] See derivation in [5] or more detailed in Appendix B.

with the density matrix

$$\rho(t) = \begin{pmatrix} \frac{1}{2}\cos^2\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin\left(\frac{gt}{\hbar}\right)\cos\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin\left(\frac{gt}{\hbar}\right)\cos\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\cos^2\left(\frac{gt}{\hbar}\right) \\ \frac{1}{2}\sin\left(\frac{gt}{\hbar}\right)\cos\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin^2\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin^2\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin\left(\frac{gt}{\hbar}\right)\cos\left(\frac{gt}{\hbar}\right) \\ \frac{1}{2}\sin\left(\frac{gt}{\hbar}\right)\cos\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin^2\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin^2\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin\left(\frac{gt}{\hbar}\right)\cos\left(\frac{gt}{\hbar}\right) \\ \frac{1}{2}\cos^2\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin\left(\frac{gt}{\hbar}\right)\cos\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\sin\left(\frac{gt}{\hbar}\right)\cos\left(\frac{gt}{\hbar}\right) & \frac{1}{2}\cos^2\left(\frac{gt}{\hbar}\right) \end{pmatrix}, \quad (2.3.2)$$

with spectrum $\{1, 0, 0, 0\}$.

Now, we decide to diagonalize the density matrix and the Hamiltonian to make a change in the TPS: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H} = \mathcal{H}'_A \otimes \mathcal{H}'_B$. The density matrix at initial time is diagonalized into $\rho'(0) = |\psi(0)\rangle\langle\psi(0)|$ and the Hamiltonian H into H'. The diagonalization of the Hamiltonian was performed is section 2.2.1 and we know that keeping the same TPS and diagonalizing the Hamiltonian is equivalent to keeping the same Hamiltonian and changing the TPS. The time evolution of the system is

$$|\psi'(t)\rangle = e^{-itH'/\hbar}|\psi'(0)\rangle = \begin{pmatrix} \frac{e^{-igt/\hbar}}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (2.3.3)$$

Its density matrix is

$$\rho'(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.3.4)$$

which is independent of time. After diagonalization, $\rho(t) = \rho'(t)$.

If we diagonalize the system at the end of the unitary evolution, we can obtain the same system as if the diagonalization was made before the unitary evolution (up to rearrangement of the eigenvalues). Therefore, it does not matter to make the change in the TPS at the beginning or at the end of the time evolution process.

Chapter 3

The measurement problem

A measurement is an intervention on a quantum system which is carried out by a measuring device and has an impact on the device. The measurement process randomly gives a value about the measured observable and it seems that only one value is obtained.

3.1 History

There are disagreements and debates among scientists about the signification of quantum physics and about how the mathematics have to be interpreted in our world. In 1935, Einstein, Podolsky and Rosen (EPR) published an article where it was argued that quantum physics was an incomplete theory [20]. In this thought experiment, two quantum correlated particles at distance are considered. This paper introduced the notion of entanglement for the first time: an entangled pair of particles can be called an EPR pair. If one carries out a measurement on, say, the position of the first particle, the second particle is instantaneously affected, and the same phenomenon is obtained when measuring the momentum of the particle. However, EPR disagreed with this action at distance and claimed that the position of the second particle was determined in advance. The argument to say that quantum mechanics is incomplete was the following:

"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity." [20]

They emphasized that in every complete physical theory, each element of the theory corresponds to an element of reality. Taking the example of the two particles, if the position of the second particle is determined in advance, its momentum cannot be determined in advance because those observables are incompatible (Heisenberg uncertainty principle): therefore those observables have no physical reality at the same time and something is missing to fully describe quantum mechanics. It would then be possible to construct a better theory containing those elements of reality which had to be local. Einstein's idea was to introduce local hidden variables but Bohr later refuted it [26]. He argued that two incompatible observables could not be measured at the same time because of the choice made in the measurement apparatus, and not an absence of physical reality in one of them. Bell later showed that quantum mechanics could not be viewed as a theory with local hidden variables and therefore the system considered by EPR could not possess local hidden variables [27]. The violation of Bell's inequalities shows that quantum mechanics cannot accept local hidden variables and this result has been experimentally verified [28, 29, 30].

3.2 Description

Quantum measurement is a process that remains not perfectly understood. A measurement gives a value for some observables thus providing information about the state of a system. These observables can be for example position, energy, or momentum.

A closed quantum state evolves in an unitary fashion according to Schrödinger's equation. Furthermore, the state can be in a superposition of many different states, as Schrödinger's equation describing the evolution of quantum systems is linear. When a measurement is made, an apparatus is coupled to

the system in superposition. There is a point in the process where the apparatus gives a single outcome which is not in superposition. The measurement tells us in what particular state the system is in. Thus, the wave function resulting from a measurement is no longer in a superposition, it has evolved into one definite state. After a measurement, the evolution of the wave function is based on the evolution of the measured state. The measurement problem has been divided in three different problems by Maudlin [8]: a problem related to the fact that a measurement leads to a determinate outcome, a problem with the fact that different results can be obtained with the same wave function, and a problem that the information about a measurement has to be carried if more than one measurement happen: after a first measurement is made we know more about the outcome of a second measurement than before the first measurement took place.

More generally, the measurement problem is that the dynamics of quantum systems is divided in two parts according to the main theory: a unitary linear evolution of the system, and projections due to the measurement. The goal to solve the measurement problem would be to reduce quantum theory to only one kind of dynamics.

3.3 Approaches to the measurement problem

There exist different approaches to the measurement problem, and different ways to interpret and understand the nature of a quantum measurement [31]. The different approaches represent different schools of thought about how to interpret the mathematical results of quantum mechanics. Some of them even include new equations, and the evolution of the system is not only determined by Schrödinger's equation. Depending on the interpretation, quantum mechanics has hidden (or additional) variables [10, 11], a deterministic [32] or not [33, 34] dynamics. The reality of the wave function is also debated, since some interpretations consider the wave function to be a mathematical tool, and some other consider that it has a physical reality. This section describes a selection of those approaches and shows in what extent they have an impact in the understanding of quantum measurement.

The Copenhagen interpretation and collapse theories

The main theory[1] about quantum mechanics, the one we learn in textbooks and in quantum mechanics courses is the Copenhagen interpretation. This interpretation is the oldest one and the most widely accepted among physicists [35, 36]. It has been mainly developed by Heisenberg and Bohr. This theory considers for example that the wave function is a mathematical entity, as it gives a probability distribution for the outcome of a measurement on the system but is not a physical entity and does not have any physical reality. The wave function ψ fully describes a system and the probability of each outcome is directly readable from the wave function using Born's rule [37]. The measurement is carried out by an external observer, or a measurement apparatus: the observer is not part of the studied quantum system, it only makes observations on it. There is a cut between the system and the observer, the Heisenberg cut [38] (however there were disagreements between Bohr and Heisenberg about the cut). It separates the quantum evolution of the quantum system and the classical evolution of the observer. The evolution of the wave function is twofold: a unitary continuous deterministic change according to Schrödinger's equation and a discontinuous change brought by the external observer. According to this theory, when performing a measurement over a state in superposition, a collapse happens and the wave function reduces to only one of the possible outcomes [2]:

$$\sum_i \alpha_i \left|\phi_i\right\rangle \xrightarrow{\text{collapse}} \left|\phi_i\right\rangle . \tag{3.3.1}$$

The wave function reduces to a single outcome after interacting with a measurement apparatus. The probability of obtaining $|\phi_i\rangle$ given by the Born's rule is $|\alpha_i|^2$.

The problem with this theory is that the collapse remains completely unspecified. The collapse means that the wave function has changed instantaneously and suddenly in a non-linear fashion and therefore does not agree with Schrödinger's equation. It is considered as a "black box": a system with an input and an output but whose operation is unknown. Thus, tacitly it assumes that the dynamics is not only of one kind.

[1] Usually, the term "interpretation" is used to refer to actual different theories. Calling them theories assumes that different equations are consider and is a better denomination.

The collapse in the Copenhagen interpretation is not a physical process but only a mathematical operation, the collapse exists because an observer outside from the quantum system has made an observation. There are theories referring to a collapse as a physical process (Ghirardi–Rimini–Weber theory [39], continuous spontaneous localization model [40], Diósi–Penrose model [41]). Each of them has its particularities. It is assumed that in addition to the Schrödinger's evolution of the wave function, there is a stochastic behavior localizing the wave function in space. A collapse then occurs at a random moment in time and space and the system is thus not evolving with Schrödinger's equation at some moments. The stochastic evolution is considered to be negligible for small quantum systems and therefore the linear evolution dominates.

Hidden variables theories

In the Copenhagen interpretation it is assumed that the wave function is complete, and contains all available information about the system. However, there exist theories assuming that some degrees of freedom are not visible from the wave function: those are hidden variables theories.

The first of them is the pilot-wave theory or Bohmian mechanics developed by de Broglie in 1927 and modernized by Bohm in 1952 [10, 11]. In this theory, it is assumed that there is no collapse of the wave function and that the wave function describes the entire universe. Additionally to Schrödinger's equation describing the unitary evolution of the wave function, there is a second equation describing the evolution of the position of the state. For a given system, there is a configuration q with coordinates q^k in a configuration space Q. The coordinates of the configuration evolve according to the guide equation

$$m_k \frac{dq_k}{dt}(t) = \hbar \nabla_k \mathrm{Im} \ln \psi(q,t) = \frac{m_k j_k}{\psi^* \psi}, \tag{3.3.2}$$

where j_k is the probability current.

The hidden degrees of freedom have an influence on the outcome of the measurement, for example, the position of the particle determines the outcome of measuring a variable on the system. Those hidden variables are non-local because local hidden variables theories have been ruled out by Bell's theorem [27]. One problem with hidden variable theories are the hidden variables: we have to assume that the wave function is not complete.

Everett's many-world interpretation

Everett's many-world interpretation [9] main advantage is that there is no collapse of the wave function during a measurement process. Everett quantum mechanics considers the observer to be part of a larger quantum system, and defines a wave function for the universe containing all the information about the universe. In Everettian quantum mechanics, the wave function evolves unitarily everywhere in the universe and a measurement process is also considered to be unitary. Therefore, contrarily to the Copenhagen interpretation, the dynamics of the wave function is not twofold anymore: only the unitary evolution is preserved.

The idea is that the collapse of the wave function can be abandoned using relative systems. Considering a system S with subsystems S_A and S_B, each substate in S_A exists relatively to a substate in S_B. The correspondence can be made with a quantum system and an observer: there exist different substates for the observer, each of them correlated to a different substate of the quantum system. A total wave function representing the quantum system S_A and the observer could be written:

$$|\Psi\rangle = \sum_i |\phi_i\rangle^{S_A} |O_i\rangle^{S_B}, \tag{3.3.3}$$

where $|O_i\rangle$ represent all the possible different outcomes. The environment, the rest of the universe, can be added to the total wave function and is also correlated to the quantum system and the observer. There is no collapse, the wave function branches and each outcome occurs in a different "world". The branches are non-interacting. Each time a measurement is made, every outcome happens but the observer only stays in one of the created worlds as he is correlated to one of the substates of the quantum system. Before a measurement, the observer cannot be in a superposition of several outcomes because we cannot experience superposition. Therefore, in Everettian quantum mechanics, a superposition of states represents independent observers experiencing a measurement process and obtaining different results in different worlds.

The idea of many-world developed from relative states was brought by DeWitt [42]. The theory was first not taking into account probabilities as each world was equally likely to exist, but there have been

improvements and the Born's rule can now be re derived from Everettian quantum mechanics [43]. This is accomplished by considering that the observer can exactly know the state of the universe they are in, without knowing the branch they are in.

However, this theory cannot be confirmed by any experiment (the same can be said about Copenhagen interpretation). Indeed, the experimental results from the many-world interpretation that can be recovered in a lab are the same as those from the Copenhagen interpretation. Moreover, the many-world theory assumes that each universe is independent from the others, excluding the possibility of observing them.

Other theories

Some other theories of quantum mechanics have to be mentioned. As an example, the quantum bayesianism or Qbism [44, 45], where the wave function is not an element of reality but represents the degrees of belief an agent has on a system. It is based on Bayesian probabilities. A measurement process is viewed as the update of the degrees of belief the agent has on the system. When several agents make the same measurement, they can measure the same outcome but they individually update their own belief about the system. A measurement apparatus is an extension of the agent.

Another theory is relational quantum mechanics developed by Carlo Rovelli [33]. The quantum systems are not described independently but by the relations that exist between them. A measurement is then a physical interaction and the observers and observed become correlated with respect to each other. When a measurement is done, the observers and observed are actualized.

One can also mention other theories such as consistent histories [46], the transactional interpretation [47], among many others.

3.4 Measurement on a qubit

We consider a two-qubit state

$$|\psi\rangle = p\,|0\rangle + e^{i\alpha}\sqrt{1-|p|^2}\,|1\rangle\ , \quad 0 \leq p \leq 1\,, \tag{3.4.1}$$

with $p \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ in a Hilbert space $\mathcal{H}_A$ with dimension 2.

We make the state and a measurement apparatus M evolve together:

$$|\psi\rangle \to |\Psi\rangle = p\,|00\rangle + e^{i\alpha}\sqrt{1-|p|^2}\,|11\rangle\ , \tag{3.4.2}$$

which gives a state vector $|\Psi\rangle$ in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with dimension 4.

The density matrix for $|\Psi\rangle$ is

$$\rho = |p|^2\,|00\rangle\langle 00| + pe^{-i\alpha}\sqrt{1-|p|^2}\,|00\rangle\langle 11| + e^{i\alpha}\sqrt{1-|p|^2}p^*\,|11\rangle\langle 00| + |1-|p|^2|\,|11\rangle\langle 11|\ . \tag{3.4.3}$$

with von Neumann entropy

$$S = -\mathrm{Tr}[\rho \ln \rho] = 0\,, \tag{3.4.4}$$

since it is a pure state. The entanglement entropy is given using the reduced density matrices $\rho_A = \mathrm{Tr}_B\rho$ and $\rho_B = \mathrm{Tr}_A\rho$

$$S_{ent} = -\mathrm{Tr}[\rho_A \ln \rho_A] = -\mathrm{Tr}[\rho_B \ln \rho_B] = -|p|^2 \ln |p|^2 - (1-|p|^2)\ln(1-|p|^2)\,. \tag{3.4.5}$$

When performing a measurement, if the apparatus shows 1 (0), we can conclude that the system is in state 1 (0). The state exhibits quantum correlations.

When doing a measurement, the state collapses (or the wave function branches, or the belief of the observer updates) to a normalized state:

$$|\Psi\rangle \to |00\rangle\,, \quad \text{with probability } |p|^2\,, \tag{3.4.6}$$

with density matrix

$$\rho_{|00\rangle} = |00\rangle\langle 00|\ . \tag{3.4.7}$$

This state has entanglement entropy 0. A measurement on the system makes it collapse to a pure non-entangled state. Entanglement entropy decreases during the collapse. However, as in the Maxwell's

demon thought-experiment, we could imagine that there is another kind of entropy, namely information entropy, that has to be taken into account when the measurement is done. We talk more about the entropy in a measurement process in Chapter 5.

Chapter 4

A new approach to quantum measurement

We have seen in Chapter 3 that different theories and interpretations of the measurement problem exist: the collapse of the wave function, the branching of it creating new worlds, the update of an agent's belief in a system, among others. In this chapter, we will take the first few steps towards a radical new view of quantum measurements.

We consider that a measurement leads to a change of TPS $\mathcal{T}$ in the global Hilbert space $\mathcal{H}$ of a system with total wave function $|\Phi\rangle$ describing the measurement apparatus and the physical system being measured. As we have seen in Chapter 2, a change of TPS corresponds to a unitary transformation in the global Hilbert space, but it also excludes simply picking up a new basis for the subsystems, namely measured system and apparatus, that corresponds to simply local unitary transformation or permutations. Therefore, by changing the TPS, the description of the quantum subsystems changes to a new one, resulting in a modification of the degrees of freedom in the subsystems.

The idea behind this is that we could interpret the collapse as if the description of the system as a factor in Hilbert space is modified during the measurement process to a new factor in Hilbert space such that one no longer observes superpositions of entangled states between the physical system and the apparatus. The new system is expressed in a new tensor product structure $\mathcal{T}'$ for the global Hilbert space $\mathcal{H}$. Thus, a measurement could be viewed as finding the tensor product structure that leads to a single outcome:

$$\begin{aligned} \mathcal{T} &\xrightarrow{\text{Measurement}} \mathcal{T}' \,, \\ |\Psi\rangle &\xrightarrow{\text{Measurement}} |\Psi\rangle' \,. \end{aligned} \tag{4.0.1}$$

Since a change in the TPS is due to a global unitary transformation, there is nothing in this approach that violates unitarity and the dynamics follows exclusively the Schrödinger's equation. This contrasts with the non-unitary nature of the collapse in the Copenhagen interpretation, but it also contrasts with the many-world interpretation, as we do not expect to have the emergence of multiple outcomes under the new TPS. We will discuss more the implications of this interpretation in Section 4.3.

Before we proceed, we should warn the reader that we have not been able to conclude this program. Even for the simplest case in which the apparatus is reduced to a single qubit, and we only consider a single qubit as the physical system to be measured, the mathematics involved exceeded our initial expectations, as we will show. Thus, below we will push the approach to its current horizon and then develop, though only conceptually, what we expect to be achieved in future work and how this new interpretation of the measurement problem differs from what has been proposed so far.

4.1 Implementation

We implement the idea using the most simple case: a 2-qubit state. As in the last chapter, we entangle a state with a measurement apparatus. We work with density matrices and make the change of TPS using unitary transformations.

Let us begin with the state

$$|\psi\rangle = \cos\alpha\,|\uparrow\rangle_A + \sin\alpha\,|\downarrow\rangle_A \,, \tag{4.1.1}$$

in the Hilbert space $\mathcal{H}_A$ with dimension 2. Using Born's rule, we easily determine that the probability of obtaining a spin-up after a measurement is $\cos^2\alpha$, and the probability of obtaining a spin-down is $\sin^2\alpha$[1].

We now consider the same measurement apparatus M as in the previous chapter and make it evolve with the state through a unitary evolution

$$|\psi\rangle \longrightarrow M|\psi\rangle = |\Phi\rangle = \cos\alpha\,|\uparrow\rangle_M\,|\uparrow\rangle_A + \sin\alpha\,|\downarrow\rangle_M\,|\downarrow\rangle_A = \begin{pmatrix} \cos\alpha \\ 0 \\ 0 \\ \sin\alpha \end{pmatrix}, \tag{4.1.2}$$

using the basis $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The overall state describing both the physical system and the measurement apparatus after their interaction belongs to $\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_A$ with dimension 4. It is our toy model. Again using Born's rule, the probability of obtaining $|\uparrow\uparrow\rangle$ is $\cos^2\alpha$, the probability of obtaining $|\downarrow\downarrow\rangle$ is $\sin^2\alpha$ and the probabilities of obtaining $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are both 0.

We write explicitly the density matrix of the state $|\Phi\rangle$ in this basis:

$$\rho = |\Phi\rangle\langle\Phi| = \begin{pmatrix} \cos^2\alpha & 0 & 0 & \cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\alpha\sin\alpha & 0 & 0 & \sin^2\alpha \end{pmatrix}. \tag{4.1.3}$$

The spectrum of ρ is $\{1,0,0,0\}$. The von Neumann entropy of this state is 0 as it is a pure state. The reduced density matrices are $\rho^A = \mathrm{Tr}_B[\rho] = \begin{pmatrix} \cos^2\alpha & 0 \\ 0 & \sin^2\alpha \end{pmatrix} = \rho_B = \mathrm{Tr}_A[\rho]$, with eigenvalues $\cos^2\alpha$ and $\sin^2\alpha$. Therefore, the entanglement entropy of state $|\Phi\rangle$ is:

$$S_{\mathrm{Ent}} = -\mathrm{Tr}[\rho^A \ln \rho^A] = -\cos^2\alpha\ln(\cos^2\alpha) - \sin^2\alpha\ln(\sin^2\alpha) \tag{4.1.4}$$

which shows a maximally entangled state ($S_{\mathrm{Ent}} = 1$ with a logarithm in base 2) when $\alpha = \pi/4$ and a minimum entangled state when one of the coefficients is 0, and therefore no superposition of states.

Following for example the Copenhagen interpretation, when a measurement is performed, the state collapses into a normalized state. The outcome spin-up, for instance, is obtained with probability $\cos^2\alpha$

$$|\Phi\rangle \xrightarrow{\textbf{Collapse}} |\Phi_C\rangle = |\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.1.5}$$

Therefore, reading a spin-up on the measurement device tells us that the system is in a spin-up position. The density matrix of this state is:

$$\rho'_{(\mathrm{i})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.1.6}$$

with (i) the Roman number for 1. The entanglement entropy is 0. It is also possible to obtain a normalized state $|\downarrow\downarrow\rangle$ with probability $\sin^2\alpha$ and with a 4×4 density matrix $\mathrm{diag}\{0,0,0,1\}$.

Diagonalizing unitarily the density matrix

We notice that in each case the resulting density matrix is a diagonal matrix. The density matrix is a Hermitian matrix. and can therefore be diagonalized using unitary matrices. As we have seen in 2.2.1, this could imply a change in the TPS if the two TPS's are not related by local unitary transformations or permutations of subsystems. Therefore, when diagonalizing a density matrix, we make a change of TPS for the global Hilbert space, which is the core idea of this new approach. We model a measurement

[1] Note that in spite of our notation refers to spin-1/2 systems, everything is applicable for general qubits with states $|0\rangle$ and $|1\rangle$.

process by diagonalizing the density matrix of the initial state in superposition toward the density matrix of a normalized state not in a superposition:

$$\rho = D\rho' D^\dagger \,, \tag{4.1.7}$$

where ρ' is the diagonalized density matrix, and D are 4×4 unitary matrices. Given that the spectrum of ρ is $\{1, 0, 0, 0\}$, there are 4 different ways to diagonalize the density matrix:

$$\text{Case (i)}, |\Phi\rangle \to |\uparrow\uparrow\rangle' \text{:}\ \rho = D_{(\text{i})}\rho'_{(\text{i})}D_{(\text{i})} \,, \tag{4.1.8a}$$

$$\text{Case (ii)}, |\Phi\rangle \to |\uparrow\downarrow\rangle' \text{:}\ \rho = D_{(\text{ii})}\rho'_{(\text{ii})}D_{(\text{ii})} \,, \tag{4.1.8b}$$

$$\text{Case (iii)}, |\Phi\rangle \to |\downarrow\uparrow\rangle' \text{:}\ \rho = D_{(\text{iii})}\rho'_{(\text{iii})}D_{(\text{iii})} \,, \tag{4.1.8c}$$

$$\text{Case (iv)}, |\Phi\rangle \to |\downarrow\downarrow\rangle' \text{:}\ \rho = D_{(\text{iv})}\rho'_{(\text{iv})}D_{(\text{iv})} \,, \tag{4.1.8d}$$

where $D_{(I)}, I \in \{\text{i, ii, iii, iv}\}$ correspond to each case. It is indeed possible, regarding the degeneracy of the eigenvalue 0, to diagonalize the initial density matrix to the density matrices corresponding to state $|\uparrow\downarrow\rangle'$ and state $|\downarrow\uparrow\rangle'$ even though those outcomes are not present in the initial state $|\Phi\rangle$. One way to interpret this is that these diagonalizing matrices not only can change the TPS, but also instantiate local unitaries that correspond to a local basis transformation, including the ones in which the apparatus is described 180 degrees rotated with respect to its original orientation. Ideally, we would like to modulo out these sort of transformations, but unfortunately we have not been able to do that during the period of the project. We will comment more about this and related issues below.

As we consider a change of TPS in the global Hilbert space, the representation of the physical system before and after the measurement process is not the same anymore. This is emphasized by a prime after each final state in equations (4.1.8), to denote that there is a change in the factor in Hilbert space describing the physical system, which remains the same. Thus, it is its attribution in Hilbert space that is changed in such a way that the physical system together with the apparatus is no longer in a superposed state.

The 4 sets of matrices that bring the density matrix into one of the four possible diagonal forms are explicitly shown in Appendix C.1. Here is shown the derivation to get the set of matrices $D_{(\text{i})}$. In this case, the density matrix is diagonalized into:

$$\rho'_{(\text{i})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.1.9}$$

We use equation (4.1.8a) in order to determine the coefficients of the matrices $D_{(\text{i})}$:

$$\begin{aligned}
&\rho'_{(\text{i})} = D_{(\text{i})}^{-1}\rho D_{(\text{i})} \Leftrightarrow D_{(\text{i})}\rho'_{(\text{i})} = \rho D_{(\text{i})} \\
&\Leftrightarrow \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & 0 & 0 & \cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\alpha\sin\alpha & 0 & 0 & \sin^2\alpha \end{pmatrix} \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \\
&\Leftrightarrow \begin{pmatrix} a & 0 & 0 & 0 \\ e & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ m & 0 & 0 & 0 \end{pmatrix} = \\
&= \begin{pmatrix} a\cos^2\alpha + m\cos\alpha\sin\alpha & b\cos^2\alpha + n\cos\alpha\sin\alpha & c\cos^2\alpha + o\cos\alpha\sin\alpha & d\cos^2\alpha + p\cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\cos\alpha\sin\alpha + m\sin^2\alpha & b\cos\alpha\sin\alpha + n\sin^2\alpha & c\cos\alpha\sin\alpha + o\sin^2\alpha & d\cos\alpha\sin\alpha + p\sin^2\alpha \end{pmatrix} .
\end{aligned} \tag{4.1.10}$$

We can conclude that:

$$\begin{cases} a \in \mathbb{C} \\ b = -n\tan\alpha \\ c = -o\tan\alpha \\ d = -p\tan\alpha \\ e = 0 \\ f \in \mathbb{C} \\ g \in \mathbb{C} \\ h \in \mathbb{C} \end{cases} \begin{cases} i = 0 \\ j \in \mathbb{C} \\ k \in \mathbb{C} \\ l \in \mathbb{C} \\ m = a\tan\alpha \\ n \in \mathbb{C} \\ o \in \mathbb{C} \\ p \in \mathbb{C} \end{cases} \Rightarrow D_{(\mathrm{i})} = \begin{pmatrix} a & -n\tan\alpha & -o\tan\alpha & -p\tan\alpha \\ 0 & f & g & h \\ 0 & j & k & l \\ a\tan\alpha & n & o & p \end{pmatrix}. \quad (4.1.11)$$

The derivations for the other cases are shown in Appendix C.1. The other matrices are:

$$\text{Case (ii): } D_{(\mathrm{ii})} = \begin{pmatrix} -m\tan\alpha & b & -o\tan\alpha & -p\tan\alpha \\ e & 0 & g & h \\ i & 0 & k & l \\ m & b\tan\alpha & o & p \end{pmatrix}, \quad (4.1.12a)$$

$$\text{Case (iii): } D_{(\mathrm{iii})} = \begin{pmatrix} -m\tan\alpha & -n\tan\alpha & c & -p\tan\alpha \\ e & f & 0 & h \\ i & j & 0 & l \\ m & n & c\tan\alpha & p \end{pmatrix}, \quad (4.1.12b)$$

$$\text{Case (iv): } D_{(\mathrm{iv})} = \begin{pmatrix} -m\tan\alpha & -n\tan\alpha & -o\tan\alpha & d \\ e & f & g & 0 \\ i & j & k & 0 \\ m & n & o & d\tan\alpha \end{pmatrix}, \quad (4.1.12c)$$

where $a, \ldots, p$ are complex numbers. Note that the matrices depend on the initial state, which is parametrized by α. This is crucial since later on we conjecture how the Born rule can be derived in terms of these matrices. Each of the matrices contains 10 independent complex parameters and therefore 20 real parameters. A 4×4 complex matrix contains a maximum of $4 \times 4 \times 2 = 32$ independent real parameters. Imposing the condition of diagonalization lowers this number to 20, but we still have to impose that the matrices are unitary. We consider for simplicity that they are special unitary and neglect the global phase. They belong to the matrix group $SU(4)$ which has a maximum of 15 independent real parameters. Imposing unitarity on those matrices is going to lower the number of independent real parameters under 15.

In order to impose unitarity we consider the Euler angle parametrization for $SU(4)$ [48]. A special unitary matrix $U \in SU(4)$ can be written in terms of Euler angles:

$$U = e^{i\lambda_3\alpha_1} e^{i\lambda_2\alpha_2} e^{i\lambda_3\alpha_3} e^{i\lambda_5\alpha_4} e^{i\lambda_3\alpha_5} e^{i\lambda_{10}\alpha_6} e^{i\lambda_3\alpha_7} e^{i\lambda_2\alpha_8} e^{i\lambda_3\alpha_9} e^{i\lambda_5\alpha_{10}} e^{i\lambda_3\alpha_{11}} e^{i\lambda_2\alpha_{12}} e^{i\lambda_3\alpha_{13}} e^{i\lambda_8\alpha_{14}} e^{i\lambda_{15}\alpha_{15}}, \quad (4.1.13)$$

where $\alpha_1, \alpha_2, \ldots \alpha_{15}$ are real parameters and λ_i are the generators of $SU(4)$. They are the equivalent in 4 dimensions of the Pauli matrices in 2 dimensions and Gell-Mann matrices in 3 dimensions. We can write this matrix explicitly (Appendix C.2 and Appendix C.3) and visualise it using the Mathematica notebook in Appendix D. We then impose that the unitary matrix obtained takes the form of the $D_{(I)}$ matrices by tuning the α_i in equation (4.1.13). We, therefore, obtain matrices obeying the two conditions: they diagonalize the density matrix into one of the given forms and they are unitary.

One possible parametrization of $D_{(I)}$ was obtained by inspection:

$$\text{Case (i): } \alpha_1 = \alpha_3 + \alpha_5 + 2\alpha_7, \alpha_2 = \pi/2, \alpha_6 = \pi/2, \alpha_8 = -\alpha, \alpha_{12} = \pi/2\,, \quad (4.1.14a)$$

$$\text{Case (ii): } \alpha_1 = \alpha_3 + \alpha_5 + 2\alpha_7, \alpha_2 = \pi/2, \alpha_6 = \pi/2, \alpha_8 = -\alpha, \alpha_{12} = 0\,, \quad (4.1.14b)$$

$$\text{Case (iii): } \alpha_1 = \alpha_3 + \alpha_5 + 2\alpha_7, \alpha_2 = \pi/2, \alpha_6 = \pi/2, \alpha_8 = \pi/2 - \alpha, \alpha_{10} = \pi/2\,, \quad (4.1.14c)$$

$$\text{Case (iv): } \alpha_1 = -\alpha_3 - \alpha_5, \alpha_2 = 0, \alpha_4 = 0, \alpha_6 = \pi/2 - \alpha\,. \quad (4.1.14d)$$

For cases (i), (ii) and (iii), there are still 10 free parameters. Those can be changed without any impact on the unitary and diagonalizing conditions imposed on the matrices. In case (iv) there are still 11 free parameters. No explanation for the difference in the number of parameters has been found yet. It

does not seem possible to increase the number of free parameters for cases (i), (ii) and (iii) using this parametrization. At this stage, it is not clear if this parametrization exhausts the set of matrices for each case. This is something that needs more investigation in the future, in particular for our conjectured understanding on how the Born rule could be recovered (see next section).

Recovering Born's rule

The initial state considered is

$$|\Phi\rangle = \cos\alpha\,|\uparrow\uparrow\rangle + \sin\alpha\,|\downarrow\downarrow\rangle\,. \tag{4.1.15}$$

With Born's rule, we expect to obtain, after a measurement, the normalized state $|\uparrow\uparrow\rangle$ with probability $\cos^2\alpha$ or the normalized state $|\downarrow\downarrow\rangle$ with probability $\sin^2\alpha$. The probability of obtaining $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ is zero. The probability obtained using the Born's rule is empirically verified and should be recovered in any interpretation of quantum mechanics. Thus, the approach we present here should give the same results, and can only be considered as a new interpretation of the quantum measurement if probabilities can be recovered from its formalism.

We would now like to recover a notion of probability directly coming from the matrices $D_{(I)}$. The first idea is to make use of the Haar measure which can be interpreted as the volume element for a matrix group. The aim is to compare the volume of each set of $D_{(I)}$ matrices and recover a notion of probabilities from their normalized ratio. There is an infinite number of matrices in each of these sets, but we still would like to show that the infinity of matrices bringing the density matrix to a density matrix corresponding to $|\uparrow\uparrow\rangle'$ or $|\downarrow\downarrow\rangle'$ "is bigger" than the infinity of those bringing the density matrix to a state $|\uparrow\downarrow\rangle'$ or $|\downarrow\uparrow\rangle'$ for our initial state in equation (4.1.15). More precisely, the ratio between the size of these sets should recover the expected probability relations.

Since the sets are subsets of $U(4)$, we start by considering the Haar measure for $SU(4)$ (which only differs by the measure associated with a global phase from $U(4)$) [48]:

$$dV_{SU(4)} = \cos(\alpha_4)^3\cos(\alpha_6)\cos(\alpha_{10})\sin(2\alpha_2)\sin(\alpha_4)\sin(\alpha_6)^5\sin(2\alpha_8)\sin(\alpha_{10})^3\sin(2\alpha_{12})d\alpha_{15}\dots d\alpha_1\,. \tag{4.1.16}$$

However, we have seen that some parameters are fixed for each set of matrices. In each case, if we replace the parameter by its value directly in the Haar measure, we get a measure zero and therefore the size of each set would be zero. The right approach would then be to define the measure for the subset obtained when the parameters are fixed from the beginning. We would therefore use a measure for each of the sets and not use the Haar measure for $SU(4)$. Let's start by considering a much more simplified case to build some intuition.

Geometric Incursion: In a sphere with volume $V = \frac{4}{3}\pi R^3$, finding the probability of picking a point on the equatorial disc of the sphere of area $A = \pi R^2$ rather than everywhere else in the sphere is equivalent to the ratio of the area of the disc over the volume of the sphere. Because the sphere is a 3-dimensional object and the disc a 2-dimensional object, the probability should be equal to 0 since the area of the disc defines a submanifold with measure zero in relation to the manifold defined by the sphere.

Consider the spherical coordinates

$$\begin{cases} x = lr\cos\theta\sin\phi \\ y = lr\cos\theta\cos\phi \\ z = lr\sin\theta\,, \end{cases} \tag{4.1.17}$$

where we introduced the non-dimensional parameter l which helps us to keep track of the dimension of the manifold. The volume element for a sphere in three dimensions is

$$dV_{SO(3)} = l^3r^2\sin\theta dr d\theta d\phi\,, \tag{4.1.18}$$

which has dimension 3 since the power of the parameter l is 3. If we want to compute the area of the disc for $\theta = 0$, we get a measure 0 but that is because it defines a submanifold of the sphere. Instead, we have to compute the measure for the 2-dimensional submanifold where the coordinates are then:

$$\begin{cases} x = lr\sin\phi \\ y = lr\cos\phi \\ z = 0 \end{cases}, \tag{4.1.19}$$

giving

$$dA = l^2 r^2 dr d\phi \,. \tag{4.1.20}$$

We see that we can keep track of the dimensionality of the submanifold by looking at the power of the parameter l. The ratio of the area of the disc over the volume of the sphere of radius R is $\frac{3}{4Rl}$. If the radius of the sphere is fixed, we can recover a notion of probability by taking the limit where l goes to infinity[2]. Therefore, we define the probability of picking a point on the equatorial disk for a sphere of size R as:

$$P = \lim_{l\to\infty} \frac{\int dA}{\int dV_{SO(3)}} \,. \tag{4.1.21}$$

This simple example serves to illustrate what happens in our case. Geometrically, $SU(4)$ defines a manifold, and the fixation of some of its parameters define submanifolds. The Haar measure associated with $SU(4)$ above corresponds to its natural embedding in the flat Euclidean space $\mathbb{R}^{16}$. Once some of the parameters are fixed, we need to consider the pullback of the flat metric onto the defined submanifold by fixing such parameters. We expect that the different sets of matrices $D_{(I)}$ will define different submanifolds with different dimensions, so we need to keep track of their dimensionality to talk meaningfully about probabilities associated with them.

Let's refine our intuition with a simple example closer to what we are interested in. Consider the group $U(2)$ instead. A matrix $U \in U(2)$ is written:

$$U = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = e^{i\varphi/4} \begin{pmatrix} l e^{i\varphi_1}\cos\theta & l e^{i\varphi_2}\sin\theta \\ -l e^{-i\varphi_2}\sin\theta & l e^{-i\varphi_1}\cos\theta \end{pmatrix} , \tag{4.1.22}$$

where the dimensionless parameter l was inserted in front of each parameter. The Haar measure, up to a normalization constant (see Appendix C.4), is obtained by taking the determinant of the Jacobian matrix and is:

$$dV_{U(2)} = l^4 |\frac{1}{2}\sin 2\theta| d\varphi d\varphi_1 d\varphi_2 d\theta \,. \tag{4.1.23}$$

The same result can be obtained by taking the line element from each factor in the matrix U

$$\begin{aligned} dl^2 &= dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 \\ &= l^2(A_{11} d\varphi^2 + A_{12} d\varphi d\theta + A_{21} d\varphi d\varphi_1 + \dots) = l^2 \sum_{i,j=1}^{4} A_{ij} dq_i dq_j, \quad q_i \in \{\varphi, \theta, \varphi_1, \varphi_2\} \,. \end{aligned} \tag{4.1.24}$$

The measure for U(2) is recovered with

$$\begin{aligned} dV_{U(2)} &= l^4 \left| \det \sum_{i,j=1}^{4} A_{ij} \left|i\right\rangle \left\langle j\right| \right|^{\frac{1}{2}} d\varphi d\theta d\varphi_1 d\varphi_2 \\ &= l^4 \left| \frac{1}{2}\sin 2\theta \right| d\varphi d\varphi_1 d\varphi_2 d\theta \,. \end{aligned} \tag{4.1.25}$$

If we now decide to fix, as an example, $\theta = 0$, we would obtain a measure 0 if we simply used the measure derived for $U(2)$. However, this is analogous to using the volume element of a sphere in three dimensions, $dV_{SO(3)} = r^2 \sin\theta dr d\theta d\phi$, to compute the area of the disk for $\theta = 0$. Of course, we would get a vanishing area, but that is because we need to find the volume associated with the submanifold defined by the disk instead, which is $dA = r dr d\phi$. When $\theta = 0$, the matrix U is then:

$$U = e^{i\varphi}/4 \begin{pmatrix} l e^{i\varphi_1} & 0 \\ 0 & l e^{-i\varphi_1} \end{pmatrix} , \tag{4.1.26}$$

which gives a volume element

$$dV = \frac{1}{2} l^2 d\varphi d\varphi_1 \,, \tag{4.1.27}$$

[2] Alternatively, we could have introduced l^{-1} as a parameter, introducing a notion of lattice (the lattice coordinates are defined as $x_i = i * l$, where l is the size of the lattice and $i \in \mathbb{Z}$). Then, one can easily build the intuition that as the size of the lattice goes to zero to recover continuity, although the amount of points in the area of the disk goes to infinity, the amount of points in the volume of the sphere goes to infinity linearly faster, which is another way of saying that the area of the disk has a measure zero volume.

and represents a submanifold with dimension 2. In our case, what we need to do, is to find the appropriate volume element associated with each submanifold instead defined by the fixation of the parameters α_i for each set of matrices $D_{(I)}$.

Unfortunately, we have not been able to compute the measure associated with the manifolds defined by the $D_{(I)}$ matrices. As it is shown in Appendix C.3, each set of matrices is made of 4×4 complex matrices with several parameters. For instance, $D_{(\mathrm{i})}$ is explicitly shown below:

$$D_{(\mathrm{i})} = \begin{pmatrix} -\cos(\alpha) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & \sin(\alpha)\cos(\alpha_{10}) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} + 6\alpha_7 + 6\alpha_9\right)} \\ 0 & \sin(\alpha_{10})\sin(\alpha_4) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} - 6\alpha_5 - 12\alpha_7\right)} \\ 0 & -\sin(\alpha_{10})\cos(\alpha_4) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15}\right)} \\ -\sin(\alpha) e^{-\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & -\cos(\alpha)\cos(\alpha_{10}) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} + 6\alpha_7 + 6\alpha_9\right)} \end{pmatrix}$$

$$\begin{pmatrix} \sin(\alpha)\sin(\alpha_{10}) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & 0 \\ -\cos(\alpha_{10})\sin(\alpha_4) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} + 6(\alpha_5 + 2\alpha_7)\right)} & -e^{-\frac{1}{2} i \left(\sqrt{6}\alpha_{15} + 4\alpha_7\right)} \cos(\alpha_4) \\ \cos(\alpha_{10}) e^{\frac{i\alpha_{15}}{\sqrt{6}} - \frac{2i\alpha_{14}}{\sqrt{3}}} \cos(\alpha_4) & -e^{i\alpha_5 - i\sqrt{\frac{3}{2}}\alpha_{15}} \sin(\alpha_4) \\ -\cos(\alpha)\sin(\alpha_{10}) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & 0 \end{pmatrix}. \tag{4.1.28}$$

with $\alpha_3, \alpha_5, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13} \in [0, \pi]$, $\alpha_4, \alpha_{12} \in [0, \pi/2]$, $\alpha_{14} \in [0, \pi/\sqrt{3}]$ and $\alpha_{15} \in [0, \pi/\sqrt{6}]$.

Thus, computing its measure by the methods above basically consists of embedding the set into $\mathbb{R}^{16}$ and then computing the line element defined in that submanifold, from which the pullback metric can be read off, and its determinant calculated, finally providing the appropriate measure. So far, this calculation seems to be intractable. Nonetheless, we will conjecture how the results should look like and how the Born rule could be recovered from the measure of these sets.

For each case, the measure $dV_{(I)}$ will look like:

$$dV_{(I)} = l^{\lambda_{(I)}} f(\alpha_j, \alpha) \prod_j d\alpha_j, \quad j \in A_{(I)}\,, \tag{4.1.29}$$

with $A_{(\mathrm{i})} = A_{(\mathrm{ii})} = \{3, 4, 5, 7, 9, 10, 11, 13, 14, 15\}$, $A_{(\mathrm{iii})} = \{3, 4, 5, 7, 9, 11, 12, 13, 14, 15\}$ and $A_{(\mathrm{iv})} = \{3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. Then, the volume of each set is given by

$$V_{D_{(I)}} = \int dV_{D_{(I)}} = F_{(I)}(\alpha)\,, \tag{4.1.30}$$

and the probability of picking a matrix $D_{(I)}$

$$P_{(I)} \equiv \lim_{l \to \infty} \frac{F_{(I)}(\alpha)}{\sum_K F_{(K)}(\alpha)}\,, \tag{4.1.31}$$

where we introduced the limit where l goes to infinity since we might be comparing measures of submanifolds of different dimensions, in other words, lower-dimensional submanifolds have measure zero in comparison with higher-dimensional ones.

Once the probabilities are recovered, we expect the following results, agreeing with Born's rule:

$$\frac{P_{(\mathrm{i})}}{P_{(\mathrm{iv})}} = \frac{P_{\uparrow\uparrow}}{P_{\downarrow\downarrow}} = \cot^2\alpha\,, \quad P_{(\mathrm{ii})} = P_{\uparrow\downarrow} = 0\,, \quad P_{(\mathrm{iii})} = P_{\downarrow\uparrow} = 0\,. \tag{4.1.32}$$

The sets for $D_{(\mathrm{ii})}$ and $D_{(\mathrm{iii})}$ are expected to define lower-dimensional manifolds such that the probabilities would be zero for them. However, it could be that some of the remaining free parameters have to be fixed to get those results, defining new sets. Here, by fixing the free remaining parameters, we imply that some of the $D_{(I)}$ matrices in each set are actually not to be taken into account in this attempt to reproduce a measurement process. It could be that they represent the same matrices modulo local unitary transformations or permutations of subsystems. We thus have to track them and eliminate them: that is what we can do when tuning the free remaining parameters in order to get equations (4.1.32). We only want matrices that reproduce a measurement process as a change of TPS in the Hilbert space, such that after a measurement is done, the new factors in Hilbert space associated with apparatus and physical system result into a single state. We would then have to repeat the above procedure, but each measure would now be defined in a set $D'_{(I)} = D_{(I)}/(SU(2)^2 \times P)$ where P defines permutations and $SU(2)$ the local unitaries for each qubit.

We can draw an analogy: let us imagine a big bag full of all of the four kinds of matrices we obtained. Each of the 4 kinds possesses an infinite amount of matrices. When doing a measurement on the qubit

state $|\Phi\rangle$, we imagine that the combined description of the apparatus and physical system being measured will have their factorization in the Hilbert space reassigned randomly, as if we were selecting the new TPS out of a bag containing these $D_{(I)}$ matrices. Were we able to recover the Born rule by precisely measuring the size of each set of D matrices, then we would get out of the bag a matrix of kind $D_{(\mathrm{i})}$ with probability $\cos^2\alpha$, and a matrix of kind $D_{(\mathrm{iv})}$ with probability $\sin^2\alpha$.

What about the two other kinds of matrices, $D_{(\mathrm{ii})}$ and $D_{(\mathrm{iii})}$? We could interpret them as a reorientation of the measurement apparatus. The apparatus is isomorphic to $SU(2)$ (it only represents one qubit), therefore a reorientation of it is a local change of basis in only one of the Hilbert spaces and not a change in the TPS. It is thus not a problem that we obtain those matrices even though Born's rule predicts a probability 0: our approach is to consider that a measurement process is a change of TPS on the global Hilbert space, therefore excluding local unitaries and permutation of subsystems.

4.2 Challenges

The first challenge was to find a good parametrization for matrices in $SU(4)$. The parametrization we encountered allows us to write the D matrices using only real parameters. However, we are aware that other kinds of parametrization exist and that this one may not be the most suitable for our problem since the mathematical problem becomes rapidly intractable. A choice of parametrization can be viewed as a choice of the best system of coordinates to describe a physical system.

We still have to investigate what the matrices represent depending on their parameters. As was mentioned, it could be that some of them are related modulo local unitaries or permutation of subsystems and therefore are not to be taken into account in the total volume of matrices. The single qubit case has been studied and is available in Appendix C.5. It is a good way to see a reorientation of the measurement apparatus using a change of basis in the local Hilbert space. However, the single qubit does not represent a change of TPS as there is only one way to write a 2-dimensional Hilbert space. It seems useful to study this system as it permits to see that two different kinds of matrices diagonalizing the density matrix differently can be related by local unitary transformations.

Moreover, although the state vector we use seems to be simple, it was surprising to notice how the calculations got quickly complicated. Indeed, working with 4×4 unitary matrices is not something we are used to. In particular when considering subsets of the group and the computation of their volume, each computation involves computing the determinant of large square matrices of order n, that naively scales as $\mathcal{O}(n!)$ in a computer. Nonetheless, once the issues are better understood for the 2-qubit state, it would be interesting to look at higher dimensions.

4.3 Discussion

Let us suppose that we have succeeded and that this new approach provides a new interpretation about what a measurement is. We would now have a new understanding of a measurement and of the Born's rule. Taking the analogy of the bag, the probability of taking a matrix from the bag would be the probability of obtaining a given outcome from a measurement. To compare with the Copenhagen interpretation, this new approach consists in only one kind of dynamics: unitary, and therefore obeys Schrödinger's equation. Instead, we rely on the fact that we change the global Hilbert space and that implies that the subsystems change during the measurement process. In this approach, the measurement brings the state into one definite outcome, as it is required in quantum mechanics for a given observer.

Given that the change in TPS corresponds to a unitary transformation for the global Hilbert space, another future direction is to investigate what would be the measurement Hamiltonian associated with this measurement unitary. Meaning, considering the unitary TPS transformation U_M, we can instantiate it with a measurement Hamiltonian, H_{M}, such that $U \equiv \exp(iH_M t)$. By taking a closer look at this Hamiltonian, it may be possible to understand better the stability of quantum systems as factorizations of Hilbert spaces, that is, when their factorization does not change.

Chapter 5

Entropy and Speculations about a generalized second law during a measurement

In this chapter, we will speculate on the entropy change during a measurement within our approach. Indeed, we saw in Chapter 3 that the entanglement entropy decreases during the collapse of the wave function. However, the total entropy of a closed system does not decrease according to the second law of thermodynamics. Therefore, in this chapter we will investigate whether we can combine the entanglement entropy with information entropy in such a way that the total entropy of the system never decreases. To do so, we will draw a close analogy with the Maxwell's demon, but first we start by reminding the different kinds of entropy, and then discussing their interplay during a measurement. We warn the reader that these are preliminary considerations.

5.1 Definitions

There are several definitions and point of views one can adopt to talk about entropy. It encompasses both microscopic and macroscopic viewpoints, and is applicable for classical as well as quantum systems, but also in quantum information theory. Generally speaking, entropy has a different meaning depending on the situation it is employed.

The first kind of entropy that is encountered in physics is the thermodynamical entropy. In classical thermodynamics, the entropy is seen from a macroscopic point of view and is a function of state, depending on state variables. A function of state describes the system at equilibrium, but does not specify which path the system has taken in order to reach equilibrium. Let us consider a system in an initial state A following a path r_1 until the final state B. Entropy increases during this process to a maximal value which corresponds to the equilibrium state B. Knowing the final entropy does not give any information about the path the system has taken to arrive to B.

The change in entropy in classical thermodynamics is given by

$$dS_T = \frac{\delta Q_{rev}}{T}\,, \tag{5.1.1}$$

where δQ_{rev} is the heat transferred to the system. For a reversible process, the change in entropy is 0 at the end of the process. This is possible when the system is quasi-static, meaning it deviates infinitesimally from equilibrium. Adiabatic processes are reversible when considering ideal conditions: if the change in entropy is 0, the evolution of the system is reversible and will not violate the second law. Conversely, for a non-reversible process, entropy increase prevents the system from being reversible. Entropy evolution therefore provides insights about the time direction of the system.

Entropy can also be described microscopically. A macroscopic state can be instantiated by different microstates. These microstates correspond to states of the system that are indistinguishable under macroscopic observations. The entropy associated with a macrostate is defined by the number of microstates, Ω, giving rise to this macrostate. This encompasses the definition of entropy introduced by Boltzmann. It represents the number of ways to rearrange the system microscopically such that its macrostate remains the same. Quantitatively, the Boltzmann's entropy is given by

$$S_B = k_b \ln \Omega \,, \tag{5.1.2}$$

where $k_b = 1.381 \times 10^{-23}$ m^2 kg s^{-2} K^{-1} is the Boltzmann's constant. As an example, consider a box separated in two parts with distinguishable balls, the entropy of the system with an equal distribution of the balls between the left and the right parts is bigger than the entropy of a box with all the balls on the left side, for example. Indeed, there are many more different ways to rearrange the balls such that they are homogeneously distributed between both sides. Conversely, there is less possibilities to have every ball in one side, and no balls in the other side. Therefore, because of the second law of thermodynamics, a system at equilibrium corresponds to a system with a homogeneous distribution of balls.

Boltzmann's entropy can be understood using coarse-graining. Formally, it refers to dividing the space of microstates of a system into regions that correspond to macrostates. Each macrostate has a certain volume in the total space of states depending on its number of microstates. A large volume therefore corresponds to a large entropy. In equation (5.1.2), the volume of the region occupied by a macrostate is Ω.

In fact, Boltzmann's entropy can also be understood as arising from the Gibbs entropy,

$$S_G = -k_B \sum_i p_i \ln p_i \,, \tag{5.1.3}$$

when all the possible microstates have the same probability $p_i = 1/\Omega$ with Ω the number of microstates. Then,

$$S_G = -k_B \sum_i (1/\Omega) \ln(1/\Omega) = k_b \ln(\Omega) = S_B \,. \tag{5.1.4}$$

Yet, another notion of entropy appears in information theory, the Shannon entropy. It gives the amount of information transmitted per message [12]. It also describes the amount of "surprise" for the receiver when the message is read. The formula takes the same form as the Gibbs entropy:

$$H = -\sum_{i=1}^{n} p_i \log p_i, \quad \sum_i p_i = 1 \,, \tag{5.1.5}$$

with p_i the probability for a character to appear in a message.

In general, entropy is related to our ignorance about a system. We know less about a system in a macrostate with a high number of microstates because we do not know exactly the microstate the system is in. Conversely, we know everything about a system when the Boltzmann's entropy is 0 since this corresponds to a single microstate. When a message is transmitted, a large entropy corresponds to a large "surprise" for the receiver; the receiver had very little information about what the message would be.

Finally, as we have discussed throughout the thesis, we also have the von Neumann entropy in quantum mechanics which was initially introduced in Chapter 1.

In a closed system, all kinds of entropy have to be considered in order for the system's evolution to agree with the second law. Indeed, one can think about the Maxwell's demon scenario, where the thermodynamics entropy of the system decreases, but the information entropy increases when the demon erases its memory. We will put this scenario in parallel with the new understanding of the measurement process discussed in Chapter 4: the entanglement entropy disappears during a measurement but we can try to find an increase in the information entropy. We would like the total entropy after a measurement to be the same as before, since unitary transformation conserves entropy.

5.2 Entropy during a quantum measurement

We start by observing a parallel between classical and quantum mechanics: the classical version of von Neumann equation is the Liouville equation, and a classical version of the von Neumann entropy is the Gibbs entropy. While the Liouville evolution preserves the Gibbs entropy over time, the von Neumann evolution conserves the entanglement entropy. Thus, in each case, the classical/quantum version of the equation conserves the classical/quantum version of the entropy.

The approach to quantum measurements that we have developed in Chapter 4 was the following: we start with a system S in a Hilbert space $\mathcal{H}$ and we consider the measurement process to be a change

in the TPS $\mathcal{T}$ of the Hilbert space to a new TPS $\mathcal{T}'$, thus changing the description of the physical system considered to one where the system is not in a superposition anymore, reproducing the single outcome requirement for a system measured by a given classical observer (see Chapter 4). During this process, the entanglement entropy of the system drops from a maximal value $\ln 2$ (if we start from a maximally entangled 2-qubit state) to zero during the measurement. We would like to propose an explanation for the disappearance of the entanglement entropy so that the measurement process agrees with a generalized form of the second law of thermodynamics, in particular in light of the fact that a change of TPS corresponds to a unitary transformation of the overall Hilbert space, which is information preserving. The main idea developed here is to consider information entropy in a similar vein as it was considered to solve the paradox of the Maxwell's demon, which seemly could violate the second law of thermodynamics. We start by briefly reviewing the paradox in its simplest case to build the needed intuition.

Maxwell's demon: A demon sits on a box separated in two parts, containing $2N$ distinguishable particles in thermal equilibrium. The system composed by the demon and the box is perfectly isolated. The initial state of the box is that both sides have the same number of particles and therefore Boltzmann entropy S_{B} is maximized since the number of possible microstates is maximized to $\Omega = \binom{2N}{N}$. However, the demon is able to open a little door, without changing noticeably the energy of the box. The door does not create work in average, since the work is

$$W = Fd\,, \tag{5.2.1}$$

with d the displacement of the door which averages out when the demon closes the door. When he opens the door, the demon allows m particles to go from one side to the other side. The consequence is that the number of microstates decreases and the Boltzmann's entropy of the box as well. Therefore, the number of microstates such that there are $N-m$ particles one one side and $N+m$ on the other side is $\Omega' = \binom{2N}{N-m}$. The change in entropy is

$$\begin{aligned}
\Delta S_{\mathrm{B}} &= k_B \ln \Omega' - k_B \ln \Omega \\
&= k_B \left(\ln \left[\frac{2N!}{N!N!} \right] - \ln \left[\frac{2N!}{(N-10)!(N+10)!} \right] \right) \\
&\approx k_B \left(\ln \left[\frac{\sqrt{4\pi N} \left(\frac{2N}{e}\right)^{2N}}{2\pi N \left(\frac{N}{e}\right)^{2N+1}} \right] - \ln \left[\frac{\sqrt{4\pi N} \left(\frac{2N}{e}\right)^{2N}}{\sqrt{2\pi(N-m)} \left(\frac{N-m}{e}\right)^{N-m} \sqrt{2\pi(N+m)} \left(\frac{N+m}{e}\right)^{N+m}} \right] \right) \\
&\approx -k_B \frac{m^2}{N}\,,
\end{aligned} \tag{5.2.2}$$

where in the third line we used Stirling's approximation $N! \approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$ and we considered that $m \ll N$. Therefore, the change in Boltzmann's entropy violates the second law of thermodynamics. This problem took a long time to be solved. The solution is that when the demon chooses a particle, he does not know if the particle is on the left side or on the right side of the box. He has to measure and record this information on a notebook, for example. At the beginning of the experiment the notebook is empty. By recording the position of the particles to know which one he can let go through the door the demon acquires m bits of information. Using Landauer's principle, the minimum amount of energy needed to erase a bit of information is $k_B T \ln 2$ where T is the temperature of the environment [49]. Thus, when erasing his memory, the demon creates entropy

$$S_{\mathrm{Erase}} = m k_B \ln 2\,, \tag{5.2.3}$$

and the total change of entropy is:

$$\Delta S_{\mathrm{tot}} = \Delta S_{\mathrm{B}} + S_{\mathrm{Erase}} = m k_B \left(\ln 2 - \frac{m}{N} \right)\,. \tag{5.2.4}$$

Therefore, the change in entropy is positive in the case $m \ll N$. Thus, in this thought experiment, we have to consider different kinds of entropy so that the total entropy of the system does not violate the second law of thermodynamics.

Quantum Measurements: In our approach, we consider, using an analogy, that we randomly pick a matrix $D_{(I)}$ out from a bag containing all possible matrices that would diagonalize the density matrix, bringing the physical description of the system to a new description of it such that it is no longer in a superposition, realizing the single outcome expected from a measurement. Similarly to the demon's scenario, the information about the new TPS has be stored somewhere, but can be erased. Take the example where a matrix $D_{(\mathrm{i})}$ is used. The system evolves as:

$$\cos\alpha\,|\uparrow\uparrow\rangle + \sin\alpha\,|\downarrow\downarrow\rangle \to |\uparrow\uparrow\rangle' \,, \tag{5.2.5}$$

where the prime denotes that the TPS has changed and therefore the description of the subsystems too. As we bring unitarily the system from its superposed state to a non superposed state, it is always possible to go backward and therefore we do not, at that point, forget in which TPS the system was expressed. However, when we are given a system not in a superposition, it is in general not possible to recover the initial description of the system. We can imagine that after the measurement happened, we discard the information encoding the change in the TPS, such that

$$|\uparrow\uparrow\rangle'\langle\uparrow\uparrow|' \to |\uparrow\uparrow\rangle\langle\uparrow\uparrow| \,. \tag{5.2.6}$$

This recovers the notion of collapse in our approach, the imposition that before and after the measurement, the physical system's description has the same representation in the Hilbert space, the same TPS. Indeed, as the demon, we have to erase information from our memory as it cannot contain an infinite amount of information, namely the information about the TPS describing the system after the measurement. We do not know anymore how to go back to the initial description of the system, and therefore, we do not know which $D_{(I)}$ matrix was used to perform the measurement. Since we consider the matrices $D_{(\mathrm{ii})}$ and $D_{(\mathrm{iii})}$ as representing a reorientation of the system apparatus, we assume that only a matrix $D_{(\mathrm{i})}$ or a matrix $D_{(\mathrm{iv})}$ could have brought the system in a non superposed state. Knowing which set of matrices was considered carries one bit of information, thus we erase a bit of information. Using Landauer's principle, this corresponds to creating entropy,

$$S_{\mathrm{Erase}} = \ln 2 \,, \tag{5.2.7}$$

such that the total change in entropy during a measurement is then 0. Naturally, we expect this to be generalized for more complicated quantum systems as this serves only as a proof of principle.

Chapter 6

Discussions

The measurement process in quantum mechanics remains a subject which is not yet fully understood. Multiple theories about quantum mechanics exist and each of them presents a new way in understanding quantum measurement. However, these theories encounter challenges such as a non-unitarity evolution or an incomplete wave function. There are debates about the viability of these theories. It is both an important and an intriguing question to understand better the nature of measurement in quantum physics as the measurement makes superposition of states vanish. A measurement provides a value about an observable, which is a physical quantity that imparts information about the physical system considered.

The idea of our approach is to make a change in the tensor product structure of the Hilbert space. The global Hilbert space containing the description of the physical system can be a factorization of lower-dimensional Hilbert spaces. Different factorizations of the Hilbert space are available for the description of a system depending on the accessible observables, and on the needs to simplify equations or obtain a more tractable description of the physical system. Two criteria were presented: a dynamical one and a kinematical one. The dynamical criterion is due to the fact that the description of the physical system has to appear local, and the kinematical one establishes a TPS that presents a quasi-classical evolution of the system. The notion of a preferred TPS could also be related to the measurement problem. According to the new approach, during a measurement, the TPS evolves to a new one, where the quantum state describing the physical system is not in a superposition anymore. A change in the TPS corresponds to a unitary transformation in the global Hilbert space and therefore only one kind of evolution of the system is considered in the approach: a unitary evolution.

Technically, the approach is based on the unitary evolution of the density matrix toward a diagonalized density matrix which represents a system not in a superposition. This requires tools and concepts from quantum information theory. For our toy model made of a 2-qubit state, we obtain four kinds of matrices making this unitary evolution possible and we aim to recover a notion of probability (Born's rule) directly from them. The idea is to find a measure for each subsets of matrices.

The apparatus described in our toy model is very simple. However, the mathematics involved exceeded our initial expectations and we did not manage to recover the Born's rule directly from the diagonalizing matrices. Perhaps a different parametrization of unitary matrices should be considered.

Once this will be achieved, it would be interesting to study higher-dimensional Hilbert spaces and try to generalize the approach to bigger systems. Also, a better understanding of the relation between the matrices in terms of local unitary transformations is required in order to determine which of the diagonalizing matrices can be interpreted as representing a measurement.

We also present preliminary considerations about the evolution of entropy during a measurement: as it was shown, entanglement entropy decreases. We therefore have, analogically to the Maxwell's demon thought experiment, to consider other kinds of entropy, such as Shannon's entropy, in order to find a generalized second law for quantum measurement.

Appendix A

Appendix for Aspects of Quantum Information

A.1 Derivation of the von Neumann equation

The von Neumann equation in quantum physics is the equivalent of the Liouville equation in classical mechanics.

By considering the time derivative of the density matrix $\rho = |\Psi\rangle\langle\Psi|$:

$$\frac{\partial\rho}{\partial t} = \partial_t(|\Psi\rangle\langle\Psi|) = (\partial_t|\Psi\rangle)\langle\Psi| + |\Psi\rangle\partial_t(\langle\Psi|)\,. \tag{A.1.1}$$

Then, using the Schrödinger's equation $\frac{1}{i\hbar}\hat{H}|\Psi(t)\rangle = \frac{\partial|\Psi(t)\rangle}{\partial t}$:

$$\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}\hat{H}|\Psi(t)\rangle\langle\Psi(t)| - \frac{1}{i\hbar}|\Psi(t)\rangle\langle\Psi(t)|\hat{H} = \frac{1}{i\hbar}[\hat{H},\rho(t)]\,. \tag{A.1.2}$$

The solution to this equation is

$$\rho(t) = U(t)\rho(0)U^\dagger(t)\,, \tag{A.1.3}$$

where $U(t) = e^{-iHt/\hbar}$, which is unitary because the eigenvalues of the Hamiltonian are real for a closed system.

A.2 Schmidt decomposition

The Schmidt decomposition is a way to write a state vector [50].

Definition 5 (Schmidt decomposition) *Let $|\psi\rangle$ be a bipartite state in $\mathcal{H} = \mathcal{H}_A\otimes\mathcal{H}_B$, with $\dim\mathcal{H}_A = d_a$ and $\dim\mathcal{H}_B = d_b$. The matrices ρ^A and ρ^B are the density matrices of the subsystems. Then, the vector $|\psi_{AB}\rangle$ can be written as a Schmidt decomposition:*

$$|\psi_{AB}\rangle = \sum_{n=1}^{k}\sqrt{p_n}\,|u_n^A, w_n^B\rangle\,, \quad p_n > 0\,, \tag{A.2.1}$$

with $k \leq \min(d_a, d_b)$ and $|u_n^A\rangle$ and $|w_n^B\rangle$ are the orthonormalised eigenvectors of ρ^A in $\mathcal{H}_A$ and ρ^B in $\mathcal{H}_B$.

Therefore, ρ^A and ρ^B have the same eigenvalues. Entanglement is basis independent. k is called the Schmidt rank of $|\psi^{AB}\rangle$.

A.3 Entropy under unitary evolution

The von Neumann entropy $S_{\text{VN}} = -\text{Tr}\left[\rho^{AB}\ln\rho^{AB}\right]$ for a state ρ^{AB} is invariant under unitary evolution.

The entropy S' after unitary evolution is:

$$S' = -\mathrm{Tr}[U\rho U^\dagger \ln(U\rho U^\dagger)]\,. \tag{A.3.1}$$

Let $e^A = U\rho U^\dagger \Rightarrow A = \ln(U\rho U^\dagger)$. Also, $U^\dagger e^A U = \rho$ and $U^\dagger e^A U = U^\dagger(\sum_n \frac{A^n}{n!})U = (\sum_n \frac{U^\dagger A^n U}{n!}) = e^{U^\dagger A U}$.

Therefore,

$$\begin{aligned} e^{U^\dagger A U} &= \rho \\ \Leftrightarrow U^\dagger A U &= \ln\rho \\ A = \ln(U\rho U^\dagger) &= U\ln\rho U^\dagger\,. \end{aligned} \tag{A.3.2}$$

And

$$\begin{aligned} S' &= -\mathrm{Tr}[U\rho U^\dagger U \ln\rho U^\dagger] \\ &= -\mathrm{Tr}[U\rho\ln\rho U^\dagger] = -\mathrm{Tr}[U^\dagger U\rho\ln\rho] \\ &= -\mathrm{Tr}[\rho\ln\rho] = S\,, \end{aligned} \tag{A.3.3}$$

where the cyclic property of the trace is used. A unitary transformation does not change the entropy of the system. Therefore, the von Neumann equation conserves the entropy.

A.4 Entropy as a function of the density matrix eigenvalues

Let's write the density matrix using the Schmidt decomposition:

$$\rho^{AB} = \sum_{n,m} \sqrt{p_n p_m}\,|u_n^A\rangle\,|w_n^B\rangle\,\langle u_m^A|\,\langle w_m^B|\,, \tag{A.4.1}$$

then,

$$\begin{aligned} S_{VN} &= -\mathrm{Tr}\left[\sum_{n,m}\sqrt{p_n p_m}\,|u_n^A\rangle\,|w_n^B\rangle\,\langle u_m^A|\,\langle w_m^B|\ln(\sum_{n,m}\sqrt{p_n p_m}\,|u_n^A\rangle\,|w_n^B\rangle\,\langle u_m^A|\,\langle w_m^B|)\right] \\ &= -\mathrm{Tr}\left[\sum_{n,m}\sqrt{p_n p_m}\delta_{nm}\delta_{nm}\,|u_n^A\rangle\,\langle u_m^A|\otimes|w_n^B\rangle\,\langle w_m^B|\ln(\sqrt{p_n p_m})\right] \\ &= -\mathrm{Tr}\left[\sum_n p_n\,|u_n^A\rangle\,\langle u_n^A|\otimes|w_n^B\rangle\,\langle w_n^B|\ln(p_n)\right] \\ &= -\sum_{i,j}\sum_n p_n\,\langle i|\,u_n^A\rangle\,\langle u_n^A|\,i\rangle\otimes\langle j|\,w_n^B\rangle\,\langle w_n^B|\,j\rangle\ln(p_n) \\ &= -\sum_n p_n \underbrace{\sum_i \langle u_n^A|\,i\rangle\,\langle i|\,u_n^A\rangle}_{\langle u_n^A|\mathbb{1}|u_n^A\rangle=1}\otimes\underbrace{\sum_j\langle w_n^B|\,j\rangle\,\langle j|\,w_n^B\rangle}_{\langle w_n^B|\mathbb{1}|w_n^B\rangle=1}\ln(p_n)\,, \end{aligned} \tag{A.4.2}$$

and therefore:

$$S_{VN} = -\sum_n p_n\ln(p_n)\,, \tag{A.4.3}$$

with p the eigenvalues of the density matrix if we consider the density matrix to be written in a basis of its eigenvectors.

Appendix B

Linear entropy

In this appendix we derive equation (2.2.14). We follow the derivation in [5] but with more details.

Consider an arbitrary factorization of the Hilbert space $\mathcal{H} = (\mathcal{H}_A \otimes \mathcal{H}_B)_{\{\theta\}}$ where θ is "a factorization", and a product state $\rho(0) = |\psi_A(0)\rangle\langle\psi_A(0)| \otimes |\psi_B(0)\rangle\langle\psi_B(0)| = \rho_A(0) \otimes \rho_B(0)$. Consider a Hamiltonian $\hat{H} = \hat{H}_{self} + \hat{H}_{int}$.

Let us first calculate the linear entanglement entropy for ρ_A. We start with a unitary evolution

$$\hat{U} = \exp\left(-i(\hat{H}_{self} + \hat{H}_{int})t\right). \tag{B.0.1}$$

Using the Zassenhaus expansion is $e^{a(X+Y)} = e^{aX}e^{aY}e^{-\frac{a^2}{2}[X,Y]}\ldots$, we obtain

$$\begin{aligned}\hat{U} &= \exp\left(-i(\hat{H}_{self} + \hat{H}_{int})t\right)\\ &= e^{-i\hat{H}_{int}t}e^{-i\hat{H}_{self}t}e^{-\frac{(-it)^2}{2}[\hat{H}_{int},\hat{H}_{self}]}e^{\mathcal{O}(t^3)}\\ =&e^{-i\hat{H}_{int}t}e^{-\frac{(-it)^2}{2}[\hat{H}_{int},\hat{H}_{self}]}e^{-i\hat{H}_{self}t}e^{\frac{it^3}{2}[\hat{H}_{self},[\hat{H}_{int},\hat{H}_{self}]]}e^{\mathcal{O}(t^3)},\end{aligned} \tag{B.0.2}$$

where we have used $AB = [A,B] + BA$. The t^3 terms can be absorbed into $\mathcal{O}(t^3)$:

$$\begin{aligned}\hat{U} &= e^{-i\hat{H}_{int}t}e^{-\frac{(-it)^2}{2}[\hat{H}_{int},\hat{H}_{self}]}e^{-i\hat{H}_{self}t}e^{\mathcal{O}(t^3)}\\ &= e^{-i\hat{H}_{int}t}e^{-\frac{(-it)^2}{2}[\hat{H}_{int},\hat{H}_{self}]}e^{-i\hat{H}_{self}t}e^{\frac{it^3}{2}[\hat{H}_{self},[\hat{H}_{int},\hat{H}_{self}]]}e^{-\frac{it^3}{2}[\hat{H}_{self},[\hat{H}_{int},\hat{H}_{self}]]}e^{\mathcal{O}(t^3)}\\ &= e^{-i\hat{H}_{int}t}e^{-\frac{(-it)^2}{2}[\hat{H}_{int},\hat{H}_{self}]}e^{-i\hat{H}_{self}t}e^{\frac{it^3}{2}[\hat{H}_{self},[\hat{H}_{int},\hat{H}_{self}]]}e^{\mathcal{O}(t^3)}\\ &= e^{-it\left(\hat{H}_{int}+\frac{it}{2}[\hat{H}_{int},\hat{H}_{self}]\right)}e^{-i\hat{H}_{self}t}e^{\mathcal{O}(t^3)}\\ &= e^{-i\hat{E}(t)t}e^{-i\hat{H}_{self}t} + \mathcal{O}(t^3),\end{aligned} \tag{B.0.3}$$

where an Identity was inserted and the Zassenhaus expansion used. The state can be written $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)$. Two other states, called self-states are defined:$\sigma_A(t) = e^{-i\hat{H}_A t}\rho_A(0)e^{i\hat{H}_A t}$ and $\sigma_B(t) = e^{-i\hat{H}_B t}\rho_B(0)e^{i\hat{H}_B t}$, and then:

$$\begin{aligned}\rho(t) &= \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)\\ &= e^{-i\hat{E}(t)t}e^{-i\hat{H}_{self}t}\hat{\rho}(0)e^{i\hat{H}_{self}t}e^{i\hat{E}(t)t}\\ &= e^{-i\hat{E}(t)t}e^{-i(\hat{H}_A+\hat{H}_B)t}\hat{\rho}(0)e^{i(\hat{H}_A+\hat{H}_B)t}e^{i\hat{E}(t)t}\\ &= e^{-i\hat{E}(t)t}e^{-i(\hat{H}_A+\hat{H}_B)t}\left(\hat{\sigma}_A(0)\otimes\hat{\sigma}_B(0)\right)e^{i(\hat{H}_A+\hat{H}_B)t}e^{i\hat{E}(t)t}\\ &= e^{-i\hat{E}(t)t}\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)e^{i\hat{E}(t)t}.\end{aligned} \tag{B.0.4}$$

This can be expanded:

$$
\begin{aligned}
\rho(t) &= e^{-i\hat{E}(t)t}\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)e^{i\hat{E}(t)t}\\
&= \left(1-i\hat{E}(t)t+\frac{(-it)^2}{2}E^2(t)\right)\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\left(1+i\hat{E}(t)t+\frac{(it)^2}{2}E^2(t)\right)\\
&= \left(\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)-i\hat{E}(t)t\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)+\frac{(-it)^2}{2}E^2(t)\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\right)\left(1+i\hat{E}(t)t+\frac{(it)^2}{2}E^2(t)\right)\\
&= \left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)+i\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\hat{E}(t)t+\frac{(it)^2}{2}\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)E^2(t) \qquad \text{(B.0.5)}\\
&- i\hat{E}(t)t\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)-\hat{E}(t)\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\hat{E}(t)(-it)^2+\frac{(-it)^2}{2}E^2(t)\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)+\mathcal{O}(t^3)\\
&= \left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)-it\left[\hat{E}(t),\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\right]+\frac{(it)^2}{2}\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)E^2(t)\\
&- \hat{E}(t)\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\hat{E}(t)(-it)^2+\frac{(-it)^2}{2}E^2(t)\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)+\mathcal{O}(t^3)\,.
\end{aligned}
$$

Note that $[A,[A,B]]=[A,AB-BA]=A(AB-BA)-(AB-BA)A=A^2B-ABA-ABA+BA^2=A^2B+BA^2-2ABA$.

Then:

$$
\rho(t)=\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)-it\left[\hat{E}(t),\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\right]+\frac{(-it)^2}{2}\left[\hat{E}(t),\left[\hat{E}(t),\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\right]\right]+\mathcal{O}(t^3)\,, \qquad \text{(B.0.6)}
$$

and:

$$
\hat{\rho}_A(t)=Tr_B[\hat{\rho}(t)]=\hat{\sigma}_A-itTr_B\left[\left[\hat{E}(t),\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\right]\right]-\frac{t^2}{2}Tr_B\left[\left[\hat{E}(t),\left[\hat{E}(t),\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\right]\right]+\mathcal{O}(t^3)\right]. \qquad \text{(B.0.7)}
$$

We have $\hat{\rho}(0)=\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)$. Using the expression for $\hat{E}(t)=\hat{H}_{int}+\frac{it}{2}\left[\hat{H}_{int},\hat{H}_{self}\right]$, in the t^2 term, we find that the only remaining term, after absorbing all the terms $\propto t$ and $\propto t^2$ in $\mathcal{O}(t^3)$ is: $\hat{H}_{int}\hat{H}_{int}\hat{\rho}(0)+\hat{\rho}(0)\hat{H}_{int}\hat{H}_{int}-2\hat{H}_{int}\hat{\rho}(0)\hat{H}_{int}=\left[\hat{H}_{int},\left[\hat{H}_{int},\hat{\rho}(0)\right]\right]$. Then,

$$
\hat{\rho}_A(t)=Tr_B[\hat{\rho}(t)]=\hat{\sigma}_A-itTr_B\left[\left[\hat{E}(t),\left(\hat{\sigma}_A\otimes\hat{\sigma}_B\right)\right]\right]-\frac{t^2}{2}Tr_B\left[\hat{H}_{int},\left[\hat{H}_{int},\hat{\rho}(0)\right]\right]+\mathcal{O}(t^3)\,. \qquad \text{(B.0.8)}
$$

The explicit form of $\hat{H}_{int}$ is $\hat{H}_{int}=\sum_{\alpha=1}^{n_{int}}\lambda_\alpha\left(\hat{A}_\alpha\otimes\hat{B}_\alpha\right)$.

By replacing and regrouping the terms $\propto t$ and $\propto t^2$ together:

$$
\begin{aligned}
\hat{\rho}_A(t)=\hat{\sigma}_A-it\sum_\alpha\lambda_\alpha Tr_B\left[\hat{A}_\alpha\hat{\sigma}_A\otimes\hat{B}_\alpha\hat{\sigma}_B-\hat{\sigma}_A\hat{A}\otimes\hat{\sigma}_B\hat{B}_\alpha\right]\\
+\frac{t^2}{2}\sum_\alpha\lambda_\alpha Tr_B\left[\left[\hat{A}_\alpha\otimes\hat{B}_\alpha,\hat{H}_{self}\right],\hat{\sigma}_A\otimes\hat{\sigma}_B\right]\\
-\frac{t^2}{2}\sum_{\alpha\beta}\lambda_\alpha\lambda_\beta Tr_B\left[\hat{A}_\alpha\otimes\hat{B}_\alpha,\left[\hat{A}_\beta\otimes\hat{B}_\beta,\hat{\rho}(0)\right]\right]+\mathcal{O}(t^3)\,. \qquad \text{(B.0.9)}
\end{aligned}
$$

Since $Tr_B\left(\hat{O}_B\hat{\rho}_B\right)=\left\langle\hat{O}_B\right\rangle$, we can condense the last expression into: $\hat{\rho}_A(t)=\hat{\sigma}_A(t)+T_A+T_2+T_3$. We obtain T_1 using the fact that $Tr(A\otimes B)=Tr(A)Tr(B)$, and also the cyclic property of the trace

$$
\begin{aligned}
T_1 &= -it\sum_\alpha\lambda_\alpha Tr_B\left[\hat{A}_\alpha\hat{\sigma}_A\otimes\hat{B}_\alpha\hat{\sigma}_B-\hat{\sigma}_A\hat{A}\otimes\hat{\sigma}_B\hat{B}_\alpha\right]\\
&= -it\sum_\alpha\lambda_\alpha\left(Tr_B\left[\hat{A}_\alpha\hat{\sigma}_A\otimes\hat{B}_\alpha\hat{\sigma}_B\right]-Tr_B\left[\hat{\sigma}_A\hat{A}\otimes\hat{\sigma}_B\hat{B}_\alpha\right]\right)\\
&= -it\sum_\alpha\lambda_\alpha\left(Tr_B\left[\hat{A}_\alpha\hat{\sigma}_A\right]Tr_B\left[\hat{B}_\alpha\hat{\sigma}_B\right]-Tr_B\left[\hat{\sigma}_A\hat{A}\right]Tr_B\left[\hat{\sigma}_B\hat{B}_\alpha\right]\right) \qquad \text{(B.0.10)}\\
&= -it\sum_\alpha\lambda_\alpha\left(\hat{A}_\alpha\hat{\sigma}_ATr_B\left[\hat{B}_\alpha\hat{\sigma}_B\right]-\hat{\sigma}_A\hat{A}Tr_B\left[\hat{\sigma}_B\hat{B}_\alpha\right]\right)\\
&= -it\sum_\alpha\lambda_\alpha\left(\left[\hat{A}_\alpha,\hat{\sigma}_A\right]\left\langle\hat{B}_\alpha^{self}\right\rangle\right)\,.
\end{aligned}
$$

Also,

$$T_2 = \frac{t^2}{2}\sum_\alpha \lambda_\alpha \left(\left[\left[\hat{A}_\alpha, \hat{H}_A \right], \hat{\rho}_A(0) \right] \langle \hat{B}_\alpha \rangle_0 + \left[\hat{A}_\alpha, \hat{\rho}_A(0) \right] \langle \hat{B}_\alpha, \hat{H}_B \rangle_0 \right) . \tag{B.0.11}$$

Let's detail the calculation of the trace over B in T_3:

$$\begin{aligned} \mathrm{Tr}_B \left[\left[\hat{A}_\alpha \otimes \hat{B}_\alpha, \left[\hat{A}_\beta \otimes \hat{B}_\beta, \hat{\rho}(0) \right] \right] \right] &= \mathrm{Tr}_B \left[\left[\hat{A}_\alpha, \otimes \hat{B}_\alpha, \hat{A}_\beta \otimes \hat{B}_\beta \hat{\rho}(0) - \hat{\rho}(0) \hat{A}_\beta \otimes \hat{B}_\beta \right] \right] \\ = \mathrm{Tr}_B \Big[\hat{A}_\alpha \hat{A}_\beta \hat{\rho}_A(0) \otimes \hat{B}_\alpha \hat{B}_\beta \hat{\rho}_b(0) - \hat{A}_\alpha \hat{\rho}_A(0) \hat{A}_\beta \otimes \hat{B}_\alpha \hat{\rho}_b(0) \hat{B}_\beta - \hat{A}_\beta \hat{\rho}_A(0) \hat{A}_\alpha \otimes \hat{B}_\beta \hat{\rho}_b(0) \hat{B}_\alpha + \\ \hat{\rho}_A(0) \hat{A}_\beta \hat{A}_\alpha \otimes \hat{\rho}_b(0) \hat{B}_\beta \hat{B}_\alpha \Big] \\ = \hat{A}_\alpha \hat{A}_\beta \hat{\rho}_A(0) \langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 - \hat{A}_\alpha \hat{\rho}_A(0) \hat{A}_\beta \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 - \hat{A}_\beta \hat{\rho}_A(0) \hat{A}_\alpha \langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 + \hat{\rho}_A(0) \hat{A}_\beta \hat{A}_\alpha \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 . \end{aligned} \tag{B.0.12}$$

Then,

$$\begin{aligned} T_3 = \frac{-t^2}{2}\sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \Big(\hat{A}_\alpha \hat{A}_\beta \hat{\rho}_A(0) \langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 - \hat{A}_\beta \hat{\rho}_A(0) \hat{A}_\alpha \langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 - \hat{A}_\alpha \hat{\rho}_A(0) \hat{A}_\beta \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 \\ + \hat{\rho}_A(0) \hat{A}_\beta \hat{A}_\alpha \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 \Big) . \end{aligned} \tag{B.0.13}$$

We now calculate the linear entanglement entropy of the system between the two subsystems, $S(\hat{\rho})_{lin} = \big(1 - Tr(\hat{\rho}^2)\big)$, because it will give a simpler expression that the von Neumann entropy. Let's expand $\hat{\sigma}_A$ using the same reasoning as for $\hat{\rho}$:

$$\begin{aligned} \hat{\sigma}_A &= \exp\left(i\hat{H}_A t\right) \hat{\rho}(0) \exp\left(-i\hat{H}_A t\right) \\ &= \hat{\rho}_A(0) - it \left[\hat{H}_1, \rho_A(0) \right] + \frac{(-it)^2}{2} \left[\hat{H}_A, \left[\hat{H}_A, \hat{\rho}_A(0) \right] \right] + \mathcal{O}(t^3) . \end{aligned} \tag{B.0.14}$$

The operator $\hat{\sigma}_A$ is pure as $\hat{\rho}(0)$ is pure. Then, its trace is 1 and $\hat{\sigma}_A^2 = \hat{\sigma}_A$. Also, as we are working to $\mathcal{O}(t^3)$, we get that $t^2 \hat{\sigma}_A(t) = t^2 \hat{\rho_A}(0) + \mathcal{O}(t^3)$.

In the trace of $\hat{\rho}_A^2$ we do not consider the term which exhibit power of time greater than 2 because they are all absorbed into $O(t^3)$. Then, T_2 ans T_3 can be disregarded. Also certain traces are 0 giving:

$$\begin{aligned} S_{lin}(\hat{\rho}_A(t)) &= 1 - Tr\left(\hat{\sigma}_A\right) - Tr(T_1^2) - Tr(\hat{\sigma}_A(t) T_3) + O(t^3) \\ &= -Tr(T_1^2) - Tr(\hat{\sigma}_A(t) T_3) + \mathcal{O}(t^3) . \end{aligned} \tag{B.0.15}$$

We need:

$$\begin{aligned} T_1^2 &= -t^2 \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \left(\left[\hat{A}_\alpha, \hat{\sigma}_A \right] \left[\hat{A}_\beta, \hat{\sigma}_A \right] \left\langle \hat{B}_\alpha^{self} \right\rangle \left\langle \hat{B}_\beta^{self} \right\rangle \right) \\ &= -t^2 \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \left(\left[\hat{A}_\alpha, \hat{\rho}_A(0) \right] \left[\hat{A}_\beta, \hat{\rho}_A(0) \right] \left\langle \hat{B}_\alpha \right\rangle_0 \left\langle \hat{B}_\beta \right\rangle_0 \right) , \end{aligned} \tag{B.0.16}$$

giving

$$\mathrm{Tr}[T_1^2] = -t^2 \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \left(\mathrm{Tr} \left[\left[\hat{A}_\alpha, \hat{\rho}_A(0) \right] \left[\hat{A}_\beta, \hat{\rho}_A(0) \right] \right] \left\langle \hat{B}_\alpha \right\rangle_0 \left\langle \hat{B}_\beta \right\rangle_0 \right) , \tag{B.0.17}$$

which can be simplified by noticing:

$$\begin{aligned} \mathrm{Tr} \left[\left[\hat{A}_\alpha, \hat{\rho}_A(0) \right] \left[\hat{A}_\beta, \hat{\rho}_A(0) \right] \right] &= \mathrm{Tr} \left[\left(\hat{A}_\alpha \hat{\rho}_A(0) - \hat{\rho}_A(0) \hat{A}_\alpha \right) \left(\hat{A}_\beta \hat{\rho}_A(0) - \hat{\rho}_A(0) \hat{A}_\beta \right) \right] \\ = \mathrm{Tr} \left[\hat{A}_\alpha \hat{\rho}_A(0) \hat{A}_\beta \hat{\rho}_A(0) - \hat{A}_\alpha \hat{\rho}_A(0) \hat{\rho}_A(0) \hat{A}_\beta - \hat{\rho}_A(0) \hat{A}_\alpha \hat{A}_\beta \hat{\rho}_A(0) + \hat{\rho}_A(0) \hat{A}_\alpha \hat{\rho}_A(0) \hat{A}_\beta \right] \\ = \mathrm{Tr} \left[\hat{A}_\alpha \hat{\rho}_A(0) \hat{A}_\beta \hat{\rho}_A(0) \right] - \mathrm{Tr} \left[\hat{A}_\beta \hat{A}_\alpha \hat{\rho}_A(0) + \hat{A}_\alpha \hat{A}_\beta \hat{\rho}_A(0) \right] + \mathrm{Tr} \left[\hat{A}_\beta \hat{\rho}_A(0) \hat{A}_\beta \hat{\rho}_A(0) \right] \\ = 2 \langle \hat{A}_\alpha \rangle_0 \langle \hat{A}_\beta \rangle_0 - \langle \left\{ \hat{A}_\alpha \hat{A}_\beta \right\}_+ \rangle_0 . \end{aligned} \tag{B.0.18}$$

Therefore,

$$\mathrm{Tr}[T_1^2] = -t^2 \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \langle \hat{B}_\alpha \rangle \langle \hat{B}_\beta \rangle \left(2 \langle \hat{A}_\alpha \rangle_0 \langle \hat{A}_\beta \rangle_0 - \langle \left\{ \hat{A}_\alpha \hat{A}_\beta \right\}_+ \rangle_0 \right) . \tag{B.0.19}$$

In T_3 there is an overall factor t^2, therefore in $\mathrm{Tr}[\hat{\sigma}_A T_3]$ we can replace $\hat{\sigma}_A$ by $\hat{\rho}(0)$:

$$\begin{aligned}
\mathrm{Tr}\left[\hat{\rho}_A(0) T_3\right] = \frac{-t^2}{2} \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \Big(\langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 \left(\mathrm{Tr}\left[\hat{\rho}_A(0) \hat{A}_\alpha \hat{A}_\beta \hat{\rho}_A(0) \right] - \mathrm{Tr}\left[\hat{\rho}_A(0) \hat{A}_\beta \hat{\rho}_A(0) \hat{A}_\alpha \right] \right) + \\
\langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 \left(\mathrm{Tr}\left[\hat{\rho}_A(0) \hat{A}_\beta \hat{A}_\alpha \right] - \mathrm{Tr}\left[\hat{\rho}_A(0) \hat{A}_\alpha \hat{\rho}_A(0) \hat{A}_\beta \right] \right) \Big) \\
= \frac{-t^2}{2} \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \left(\langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 \left(\mathrm{Tr}\left[\hat{A}_\alpha \hat{A}_\beta \hat{\rho}_A(0) \right] - \langle \hat{A}_\alpha \rangle_0 \langle \hat{A}_\beta \rangle_0 \right) + \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 \left(\mathrm{Tr}\left[\hat{A}_\beta \hat{A}_\alpha \hat{\rho}_A(0) \right] - \langle \hat{A}_\beta \rangle_0 \langle \hat{A}_\alpha \rangle_0 \right) \right) \\
= \frac{-t^2}{2} \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \left(\langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 \left(\langle \hat{A}_\alpha \hat{A}_\beta \rangle_0 - \langle \hat{A}_\alpha \rangle \langle \hat{A}_\beta \rangle_0 \right) + \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 \left(\langle \hat{A}_\beta \hat{A}_\alpha \rangle_0 - \langle \hat{A}_\beta \rangle_0 \langle \hat{A}_\alpha \rangle_0 \right) \right) .
\end{aligned} \tag{B.0.20}$$

We can now include those result in the linear entropy formula and we get:

$$\begin{aligned}
S_{lin} = t^2 \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \Big(\langle \hat{A}_\alpha \hat{A}_\beta \rangle_0 \langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 + \langle \hat{A}_\beta \hat{A}_\alpha \rangle_0 \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 \\
- \langle \hat{A}_\alpha \rangle_0 \langle \hat{A}_\beta \rangle_0 \langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 - \langle \hat{A}_\beta \rangle \langle \hat{A}_\alpha \rangle \langle \hat{B}_\beta \hat{B}_\alpha \rangle - \langle \hat{B}_\alpha \rangle \langle \hat{B}_\beta \rangle \left(\langle \left\{ \hat{A}_\alpha \hat{A}_\beta \right\}_+ \rangle_0 - 2 \langle \hat{A}_\alpha \rangle_0 \langle \hat{A}_\beta \rangle_0 \right) \Big) + \mathcal{O}(t^3) .
\end{aligned} \tag{B.0.21}$$

In the case $\alpha = \beta$, $\hat{H}_{int} = \lambda \left(\hat{A} \otimes \hat{B} \right)$, we get that the entropy is:

$$S_{lin}(\hat{\rho}_a(t)) = 2\lambda^2 t^2 \left(\left\langle \hat{A}^2 \right\rangle_0 - \left\langle \hat{A} \right\rangle_0^2 \right) \left(\left\langle \hat{B}^2 \right\rangle_0 - \left\langle \hat{B} \right\rangle_0^2 \right) . \tag{B.0.22}$$

Entanglement entropy depends on λ which is the interaction strength, this will have an effect on the evolution of entropy, but the initial non-entangled state also has a role to play here. There are variance-like terms in (B.0.22). This means that if the variance is large, the states spreads and the linear entanglement entropy grows. In Chapter 2, we wrote λ as $\lambda_{\{\theta\}}$ to emphasize that the parameter depends on the factorization of the Hilbert space.

Appendix C

Appendix for delving into the new interpretation

C.1 Obtaining the diagonalizing matrices

The explicit forms of the D matrices $D_{(\text{ii})}$, $D_{(\text{iii})}$ and $D_{(\text{iv})}$ are derived here. The calculations for $D_{(\text{i})}$ are shown in Chapter 4. We consider the density matrix of the 2-qubit state ρ form equation (4.1.3). We aim to find the $D_{(I)}$ matrices that diagonalize ρ into $\rho'_{(I)}$, $I \in \{\text{i, ii, iii, iv}\}$.

Case (ii)

We start with case (ii), where the diagonalized density matrix corresponds to the density matrix of a normalized state $|\uparrow\downarrow\rangle$:

$$\rho'_{(\text{ii})} = D^{-1}_{(\text{ii})}\rho D_{(\text{ii})} \Leftrightarrow D_{(\text{ii})}\rho'_{(\text{ii})} = \rho D_{(\text{ii})}$$

$$\Leftrightarrow \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & 0 & 0 & \cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\alpha\sin\alpha & 0 & 0 & \sin^2\alpha \end{pmatrix} \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & n & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} a\cos^2\alpha + m\cos\alpha\sin\alpha & b\cos^2\alpha + n\cos\alpha\sin\alpha & c\cos^2\alpha + o\cos\alpha\sin\alpha & d\cos^2\alpha + p\cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\cos\alpha\sin\alpha + m\sin^2\alpha & b\cos\alpha\sin\alpha + n\sin^2\alpha & c\cos\alpha\sin\alpha + o\sin^2\alpha & d\cos\alpha\sin\alpha + p\sin^2\alpha \end{pmatrix}. \tag{C.1.1}$$

We arrive at:

$$\begin{cases} a = -m\tan\alpha \\ b \in \mathbb{C} \\ c = -o\tan\alpha \\ d = -p\tan\alpha \\ e \in \mathbb{C} \\ f = 0 \\ g \in \mathbb{C} \\ h \in \mathbb{C} \end{cases} \quad \begin{cases} i \in \mathbb{C} \\ j = 0 \\ k \in \mathbb{C} \\ l \in \mathbb{C} \\ m \in \mathbb{C} \\ n = b\tan\alpha \\ o \in \mathbb{C} \\ p \in \mathbb{C} \end{cases} \Rightarrow D_{(\text{ii})} = \begin{pmatrix} -m\tan\alpha & b & -o\tan\alpha & -p\tan\alpha \\ e & 0 & g & h \\ i & 0 & k & l \\ m & b\tan\alpha & o & p \end{pmatrix}. \tag{C.1.2}$$

Case (iii)

Then, in case (iii) the diagonalized density matrix corresponds to the density matrix of a normalized state $|\downarrow\uparrow\rangle$:

$$\rho'_{(\mathrm{iii})} = D^{-1}_{(\mathrm{iii})}\rho D_{(\mathrm{iii})} \Leftrightarrow D_{(\mathrm{iii})}\rho'_{(\mathrm{iii})} = \rho D_{(\mathrm{iii})}$$

$$\Leftrightarrow \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & 0 & 0 & \cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\alpha\sin\alpha & 0 & 0 & \sin^2\alpha \end{pmatrix} \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & O & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} a\cos^2\alpha + m\cos\alpha\sin\alpha & b\cos^2\alpha + n\cos\alpha\sin\alpha & c\cos^2\alpha + o\cos\alpha\sin\alpha & d\cos^2\alpha + p\cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\cos\alpha\sin\alpha + m\sin^2\alpha & b\cos\alpha\sin\alpha + n\sin^2\alpha & c\cos\alpha\sin\alpha + o\sin^2\alpha & d\cos\alpha\sin\alpha + p\sin^2\alpha \end{pmatrix}. \tag{C.1.3}$$

We arrive at:

$$\begin{cases} a = -m\tan\alpha \\ b = -n\tan\alpha \\ c \in \mathbb{C} \\ d = -p\tan\alpha \\ e \in \mathbb{C} \\ f \in \mathbb{C} \\ g = 0 \\ h \in \mathbb{C} \end{cases} \begin{cases} i \in \mathbb{C} \\ j \in \mathbb{C} \\ k = 0 \\ l \in \mathbb{C} \\ m \in \mathbb{C} \\ n \in \mathbb{C} \\ o = c\tan\alpha \\ p \in \mathbb{C} \end{cases} \Rightarrow D_{(\mathrm{iii})} = \begin{pmatrix} -m\tan\alpha & -n\tan\alpha & c & -p\tan\alpha \\ e & f & 0 & h \\ i & j & 0 & l \\ m & n & c\tan\alpha & p \end{pmatrix}. \tag{C.1.4}$$

Case (iv)

Finally, in case (iv) the diagonalized density matrix corresponds to the density matrix of a normalized state $|\downarrow\downarrow\rangle$:

$$\rho'_{(\mathrm{iv})} = D^{-1}_{(\mathrm{iv})}\rho D_{(\mathrm{iv})} \Leftrightarrow D_{(\mathrm{iv})}\rho'_{(\mathrm{iv})} = \rho D_{(\mathrm{iv})}$$

$$\Leftrightarrow \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & 0 & 0 & \cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\alpha\sin\alpha & 0 & 0 & \sin^2\alpha \end{pmatrix} \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 0 & d \\ 0 & 0 & 0 & h \\ 0 & 0 & 0 & l \\ 0 & 0 & 0 & p \end{pmatrix} =$$

$$= \begin{pmatrix} a\cos^2\alpha + m\cos\alpha\sin\alpha & b\cos^2\alpha + n\cos\alpha\sin\alpha & c\cos^2\alpha + o\cos\alpha\sin\alpha & d\cos^2\alpha + p\cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a\cos\alpha\sin\alpha + m\sin^2\alpha & b\cos\alpha\sin\alpha + n\sin^2\alpha & c\cos\alpha\sin\alpha + o\sin^2\alpha & d\cos\alpha\sin\alpha + p\sin^2\alpha \end{pmatrix}. \tag{C.1.5}$$

We arrive at:

$$\begin{cases} a = -m\tan\alpha \\ b = -n\tan\alpha \\ c = -o\tan\alpha \\ d \in \mathbb{C} \\ e \in \mathbb{C} \\ f \in \mathbb{C} \\ g \in \mathbb{C} \\ h = 0 \end{cases} \begin{cases} i \in \mathbb{C} \\ j \in \mathbb{C} \\ k \in \mathbb{C} \\ l = 0 \\ m \in \mathbb{C} \\ n \in \mathbb{C} \\ o \in \mathbb{C} \\ p = d\tan\alpha \end{cases} \Rightarrow D_{\alpha,(\mathrm{iv})} = \begin{pmatrix} -m\tan\alpha & -n\tan\alpha & -o\tan\alpha & d \\ e & f & g & 0 \\ i & j & k & 0 \\ m & n & o & d\tan\alpha \end{pmatrix}. \tag{C.1.6}$$

Note that these matrices can also be written:

$$
\begin{aligned}
\text{(ii):}\ D_{(\text{ii})} &= \begin{pmatrix} a & n\cot\alpha & -o\tan\alpha & -p\tan\alpha \\ e & 0 & g & h \\ i & 0 & k & l \\ -a\cot\alpha & n & o & p \end{pmatrix}, \\
\text{(iii):}\ D_{(\text{iii})} &= \begin{pmatrix} a & -n\tan\alpha & o\cot\alpha & -p\tan\alpha \\ e & f & 0 & h \\ i & j & 0 & l \\ -a\cot\alpha & n & o & p \end{pmatrix}, \\
\text{(iv):}\ D_{(\text{iv})} &= \begin{pmatrix} a & -n\tan\alpha & -o\tan\alpha & p\cot\alpha \\ e & f & g & 0 \\ i & j & k & 0 \\ -a\cot\alpha & n & o & p \end{pmatrix}.
\end{aligned}
\tag{C.1.7}
$$

Their determinants are

$$
\begin{aligned}
\text{Case (i):}\ \Delta_{(\text{i})} &= \frac{a}{\cos^2\alpha}[fkp + joh + ngl - nkh - jgp - fol]. \\
\text{Case (ii):}\ \Delta_{(\text{ii})} &= \frac{b}{\cos^2\alpha}[mhk - mgl + oel - ohi + pgi - pek]. \\
\text{Case (iii):}\ \Delta_{(\text{iii})} &= \frac{c}{\cos^2\alpha}[flm - hjm + hin - eln - fip + ejp]. \\
\text{Case (iv):}\ \Delta_{(\text{iv})} &= \frac{d}{\cos^2\alpha}[gjm - fkm - gin + ekn + fio - ejo].
\end{aligned}
\tag{C.1.8}
$$

It is here not yet assumed that those matrices are unitary.

C.2 Condition of unitarity

We would like to find the conditions on the complex parameters such that the matrices D are unitary. By definition, the matrices satisfy:

$$
D^{-1} = D^\dagger\,. \tag{C.2.1}
$$

And we then equate the two parts of the equation. Using Mathematica (Appendix D), we get for the first case:

$$
\begin{pmatrix} a^* & 0 & 0 & a^*\tan\alpha \\ -n^*\tan\alpha & f^* & j^* & n^* \\ -o^*\tan\alpha & g^* & k^* & o^* \\ -p^*\tan\alpha & h^* & l^* & p^* \end{pmatrix} =
$$

$$
\begin{pmatrix}
\frac{\cos^2\alpha}{a} & 0 & \dots \\
\frac{\sin\alpha\cos\alpha(gl-hk)}{-fkp+flo+gjp-gln-hjo+hkn} & \frac{lo-kp}{-fkp+flo+gjp-gln-hjo+hkn} & \dots \\
\frac{\sin\alpha\cos\alpha(hj-fl)}{-fkp+flo+gjp-gln-hjo+hkn} & \frac{ln-jp}{fkp-flo-gjp+gln+hjo-hkn} & \dots \\
\frac{\sin\alpha\cos\alpha(fk-gj)}{-fkp+flo+gjp-gln-hjo+hkn} & \frac{kn-jo}{-fkp+flo+gjp-gln-hjo+hkn} & \dots
\end{pmatrix}
\tag{C.2.2}
$$

$$
\begin{pmatrix}
\dots & 0 & \frac{\sin\alpha\cos\alpha}{a} \\
\dots & \frac{ho-gp}{fkp-flo-gjp+gln+hjo-hkn} & \frac{\cos^2\alpha(hk-gl)}{-fkp+flo+gjp-gln-hjo+hkn} \\
\dots & \frac{hn-fp}{-fkp+flo+gjp-gln-hjo+hkn} & \frac{\cos^2\alpha(hj-fl)}{fkp-flo-gjp+gln+hjo-hkn} \\
\dots & \frac{gn-fo}{fkp-flo-gjp+gln+hjo-hkn} & \frac{\cos^2\alpha(gj-fk)}{-fkp+flo+gjp-gln-hjo+hkn}
\end{pmatrix},
$$

implying

$$
\begin{cases}
a = \cos\alpha e^{i\beta} \\
\overline{f} = e^{i\beta}(kp - lo)/(\Delta_{(\text{i})}\cos\alpha) \\
\overline{g} = e^{i\beta}(ln - jp)/(\Delta_{(\text{i})}\cos\alpha) \\
\overline{h} = e^{i\beta}(jo - kn)/(\Delta_{(\text{i})}\cos\alpha) \\
\overline{j} = e^{i\beta}(ho - gp)/(\Delta_{(\text{i})}\cos\alpha)
\end{cases}
\quad
\begin{cases}
\overline{k} = e^{i\beta}(fp - hn)/(\Delta_{(\text{i})}\cos\alpha) \\
\overline{l} = e^{i\beta}(gn - fo)/(\Delta_{(\text{i})}\cos\alpha) \\
\overline{n} = \cos\alpha e^{i\beta}(gl - hk)/\Delta_{(\text{i})} \\
\overline{o} = \cos\alpha e^{i\beta}(jh - fl)/\Delta_{(\text{i})} \\
\overline{p} = \cos\alpha e^{i\beta}(fk - gj)/\Delta_{(\text{i})}
\end{cases}.
\tag{C.2.3}
$$

Similar results can be obtained for cases (ii), (iii) and (iv). Rewriting the matrices using those new conditions ensure that they are unitary. However, a better way to impose unitary is to look for a parametrization of $SU(4)$ with real parameters.

C.3 SU(n) and parametrization with Euler angles

In this appendix, we review how the parametrization in terms of Euler angles is obtained for $SU(2)$ and just quote the result for $SU(4)$ given in [48]. It can be useful to first define what is $SU(n)$. $SU(n)$ is a matrix group where matrices belonging to this group are called special unitary matrices. That means they are unitary with a determinant equal to one:

$$SU(n) = \{U : U^\dagger U = \mathbb{1}, \det U = 1\}, \tag{C.3.1}$$

SU(n) is a Lie group and it therefore has a Lie algebra generating the elements of the group. The Lie algebra for a special unitary group with dimension n is made of $n^2 - 1$ generators which are $n \times n$ skew Hermitian matrices. For $SU(2)$, those are the Pauli matrices which are generally used as spin measurement matrices in quantum mechanics. For $SU(3)$, they are the Gell-Mann matrices used in quantum chromadynamics.

We start by looking for the parametrization of $SU(2)$ in terms of Euler angles and the generators. The generators of $SU(2)$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.3.2}$$

We have that $\sigma_i^{2p} = \mathbb{1}$ and $\sigma_i^{2p+1} = \sigma_i$.

First we consider a special unitary matrix U:

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = U^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{C.3.3}$$

therefore:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \tag{C.3.4}$$

using the condition on the determinant we have

$$|a|^2 + |b|^2 = 1 \Rightarrow \begin{cases} a = e^{i\phi_1}\cos(\beta/2) \\ b = e^{i\phi_2}\sin(\beta/2) \end{cases}. \tag{C.3.5}$$

Therefore,

$$U = \begin{pmatrix} e^{i\phi_1}\cos(\beta/2) & e^{i\phi_2}\sin(\beta/2) \\ -e^{-i\phi_2}\sin(\beta/2) & e^{-i\phi_1}\cos(\beta/2) \end{pmatrix}. \tag{C.3.6}$$

Now we verify the parametrization from [48]:

$$U = e^{-i\alpha\sigma_3/2} e^{-i\beta\sigma_2/2} e^{-i\gamma\sigma_3/2}. \tag{C.3.7}$$

We can use the expansion

$$e^{aX} = \sum_{k=0}^{\infty} = \frac{a^k}{k!} X^k, \tag{C.3.8}$$

where X is a matrix.

Therefore,

$$
\begin{aligned}
e^{-i\alpha\sigma_3/2} &= \sum_k \frac{1}{k!}\left(\frac{-i\alpha}{2}\right)^k \sigma_3^k = \sum_{k'} \frac{1}{(2k')!}\left(\frac{-i\alpha}{2}\right)^{2k'} \sigma_3^{2k'} + \sum_{k''} \frac{1}{(2k''+1)!}\left(\frac{-i\alpha}{2}\right)^{2k''+1} \sigma_3^{2k''+1} \\
&= \sum_{k'} \frac{1}{(2k')!}(-1)^{k'}\left(\frac{\alpha}{2}\right)^{2k'} \mathbb{1} + \sum_{k''} \frac{1}{(2k''+1)!}(-i)(-1)^{k''}\left(\frac{\alpha}{2}\right)^{2k''+1} \sigma_3 \\
&= \mathbb{1}\sum_{k'} \frac{(-1)^{k'}}{(2k')!}\left(\frac{\alpha}{2}\right)^{2k'} - i\sigma_3 \sum_{k''} \frac{(-1)^{k''}}{(2k''+1)!}\left(\frac{\alpha}{2}\right)^{2k''+1} \\
&= \mathbb{1}\cos(\alpha/2) - i\sigma_3 \sin(\alpha/2) \\
&= \begin{pmatrix} \cos(\alpha/2) - i\sin(\alpha/2) & 0 \\ 0 & \cos(\alpha/2) + i\sin(\alpha/2) \end{pmatrix}.
\end{aligned}
\tag{C.3.9}
$$

Similarly,

$$
\begin{aligned}
e^{-i\beta\sigma_2/2} &= \mathbb{1}\cos(\beta/2) - i\sigma_2 \sin(\beta/2)\,, \\
e^{-i\gamma\sigma_3/2} &= \mathbb{1}\cos(\gamma/2) - i\sigma_3 \sin(\gamma/2)\,.
\end{aligned}
\tag{C.3.10}
$$

Therefore, we get the matrix

$$
U = \begin{pmatrix} \cos(\beta/2)e^{-\frac{i}{2}(\alpha+\gamma)} & -\sin(\beta/2)e^{\frac{i}{2}(-\alpha+\gamma)} \\ \sin(\beta/2)e^{\frac{i}{2}(\alpha-\gamma)} & \cos(\beta/2)e^{\frac{i}{2}(\alpha+\gamma)} \end{pmatrix}. \tag{C.3.11}
$$

We recover the Euler angle parametrization from equation (C.3.6) using $\phi_1 = (\alpha+\gamma)$ and $\phi_2 = (\gamma-\alpha)$.

One can show that from the parametrization of $SU(2)$ one can obtain the one for $SU(3)$ and $SU(4)$ [51, 52, 48]. We get that, for $U \in SU(4)$:

$$
U = e^{i\lambda_3\alpha_1} e^{i\lambda_2\alpha_2} e^{i\lambda_3\alpha_3} e^{i\lambda_5\alpha_4} e^{i\lambda_3\alpha_5} e^{i\lambda_{10}\alpha_6} e^{i\lambda_3\alpha_7} e^{i\lambda_2\alpha_8} e^{i\lambda_3\alpha_9} e^{i\lambda_5\alpha_{10}} e^{i\lambda_3\alpha_{11}} e^{i\lambda_2\alpha_{12}} e^{i\lambda_3\alpha_{13}} e^{i\lambda_8\alpha_{14}} e^{i\lambda_{15}\alpha_{15}}\,, \tag{C.3.12}
$$

where we have now 15 different parameters $\alpha_i$with ranges

$$
\begin{aligned}
0 \le \alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13} \le \pi, \\
0 \le \alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12} \le \pi/2, \\
0 \le \alpha_{14} \le \pi/\sqrt{3}, \\
0 \le \alpha_{15} \le \pi/\sqrt{6}, \\
-\pi/2 \le \alpha \le 0\,.
\end{aligned}
\tag{C.3.13}
$$

We have a parametrization of $SU(4)$ in terms of Euler angles that we can use in order to impose unitary to the $D_{(I)}$ matrices. By equating the unitary matrix U with one of the 4 kinds of matrices using Mathematica, we get the following parametrizations for the matrices:

$$
\begin{aligned}
D_{(\mathrm{i})} &= e^{i\lambda_3(\alpha_3+\alpha_5+2\alpha_7)} e^{i\lambda_2\frac{\pi}{2}} e^{i\lambda_3\alpha_3} e^{i\lambda_5\alpha_4} e^{i\lambda_3\alpha_5} e^{i\lambda_{10}\frac{\pi}{2}} e^{i\lambda_3\alpha_7} e^{-i\lambda_2\alpha} e^{i\lambda_3\alpha_9} e^{i\lambda_5\alpha_{10}} e^{i\lambda_3\alpha_{11}} \\
&\quad e^{i\lambda_2\frac{\pi}{2}} e^{i\lambda_3\alpha_{13}} e^{i\lambda_8\alpha_{14}} e^{i\lambda_{15}\alpha_{15}}\,, \qquad \text{(C.3.14)} \\
D_{(\mathrm{ii})} &= e^{i\lambda_3(\alpha_3+\alpha_5+2\alpha_7)} e^{i\lambda_2\frac{\pi}{2}} e^{i\lambda_3\alpha_3} e^{i\lambda_5\alpha_4} e^{i\lambda_3\alpha_5} e^{i\lambda_{10}\frac{\pi}{2}} e^{i\lambda_3\alpha_7} e^{-i\lambda_2\alpha} e^{i\lambda_3\alpha_9} e^{i\lambda_5\alpha_{10}} e^{i\lambda_3\alpha_{11}} \\
&\quad \mathbb{1} e^{i\lambda_3\alpha_{13}} e^{i\lambda_8\alpha_{14}} e^{i\lambda_{15}\alpha_{15}}\,, \qquad \text{(C.3.15)} \\
D_{(\mathrm{iii})} &= e^{i\lambda_3(\alpha_3+\alpha_5+2\alpha_7)} e^{i\lambda_2\frac{\pi}{2}} e^{i\lambda_3\alpha_3} e^{i\lambda_5\alpha_4} e^{i\lambda_3\alpha_5} e^{i\lambda_{10}\frac{\pi}{2}} e^{i\lambda_3\alpha_7} e^{i\lambda_2(\frac{\pi}{2}-\alpha)} e^{i\lambda_3\alpha_9} e^{i\lambda_5\frac{\pi}{2}} e^{i\lambda_3\alpha_{11}} \\
&\quad e^{i\lambda_2\alpha_{12}} e^{i\lambda_3\alpha_{13}} e^{i\lambda_8\alpha_{14}} e^{i\lambda_{15}\alpha_{15}}\,, \qquad \text{(C.3.16)} \\
D_{(\mathrm{iv})} &= e^{i\lambda_3(-\alpha_3-\alpha_5)} \mathbb{1} e^{i\lambda_3\alpha_3} \mathbb{1} e^{i\lambda_3\alpha_5} e^{i\lambda_{10}(\frac{\pi}{2}-\alpha)} e^{i\lambda_3\alpha_7} e^{i\lambda_2\alpha_8} e^{i\lambda_3\alpha_9} e^{i\lambda_5\alpha_{10}} e^{i\lambda_3\alpha_{11}} \\
&\quad e^{i\lambda_2\alpha_{12}} e^{i\lambda_3\alpha_{13}} e^{i\lambda_8\alpha_{14}} e^{i\lambda_{15}\alpha_{15}}\,. \qquad \text{(C.3.17)}
\end{aligned}
$$

Note that we can also replace α_8 by $\pi - \alpha$ in cases (i), (ii) and (iii). And α_6 by $3\pi/2 - \alpha$ in case (iv).

We obtain for case (i): $\alpha_1 = \alpha_3 + \alpha_5 + 2\alpha_7, \alpha_2 = \pi/2, \alpha_6 = \pi/2, \alpha_8 = -\alpha, \alpha_{12} = \pi/2$

$$D_{(\mathrm{i})} = \begin{pmatrix} -\cos(\alpha) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & \sin(\alpha)\cos(\alpha_{10}) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} + 6\alpha_7 + 6\alpha_9\right)} & \sin(\alpha)\sin(\alpha_{10}) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & 0 \\ 0 & \sin(\alpha_{10})\sin(\alpha_4) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} - 6\alpha_5 - 12\alpha_7\right)} & -\cos(\alpha_{10})\sin(\alpha_4) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} + 6(\alpha_5 + 2\alpha_7)\right)} & -e^{-\frac{1}{2} i \left(\sqrt{6}\alpha_{15} + 4\alpha_7\right)} \cos(\alpha_4) \\ 0 & -\sin(\alpha_{10})\cos(\alpha_4) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15}\right)} & \cos(\alpha_{10}) e^{\frac{i\alpha_{15}}{\sqrt{6}} - \frac{2i\alpha_{14}}{\sqrt{3}}} \cos(\alpha_4) & -e^{i\alpha_5 - i\sqrt{\frac{3}{2}}\alpha_{15}} \sin(\alpha_4) \\ -\sin(\alpha) e^{-\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & -\cos(\alpha)\cos(\alpha_{10}) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} + 6\alpha_7 + 6\alpha_9\right)} & -\cos(\alpha)\sin(\alpha_{10}) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & 0 \end{pmatrix} \tag{C.3.18}$$

For case (ii): $\alpha_1 = \alpha_3 + \alpha_5 + 2\alpha_7, \alpha_2 = \pi/2, \alpha_6 = \pi/2, \alpha_8 = -\alpha, \alpha_{12} = 0$

$$D_{(\mathrm{ii})} = \begin{pmatrix} \sin(\alpha)\cos(\alpha_{10}) e^{\left(\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} + 6\alpha_7 + 6\alpha_9\right)\right)} & \cos(\alpha) e^{\left(-\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)\right)} & \sin(\alpha)\sin(\alpha_{10}) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & 0 \\ \sin(\alpha_{10})\sin(\alpha_4) e^{\left(\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} - 6\alpha_5 - 12\alpha_7\right)\right)} & 0 & -\cos(\alpha_{10})\sin(\alpha_4) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} + 6(\alpha_5 + 2\alpha_7)\right)} & -e^{-\frac{1}{2} i \left(\sqrt{6}\alpha_{15} + 4\alpha_7\right)} \cos(\alpha_4) \\ -\sin(\alpha_{10})\cos(\alpha_4) e^{\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15}\right)} & 0 & \cos(\alpha_{10}) e^{\frac{i\alpha_{15}}{\sqrt{6}} - \frac{2i\alpha_{14}}{\sqrt{3}}} \cos(\alpha_4) & -e^{i\alpha_5 - i\sqrt{\frac{3}{2}}\alpha_{15}} \sin(\alpha_4) \\ -\cos(\alpha)\cos(\alpha_{10}) e^{\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} + 6\alpha_7 + 6\alpha_9\right)} & \sin(\alpha) e^{\left(-\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)\right)} & -\cos(\alpha)\sin(\alpha_{10}) - e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & 0 \end{pmatrix} \tag{C.3.19}$$

For case (iii): $\alpha_1 = \alpha_3 + \alpha_5 + 2\alpha_7, \alpha_2 = \pi/2, \alpha_6 = \pi/2, \alpha_8 = \pi/2 - \alpha, \alpha_{10} = \pi/2$

$$D_{(\mathrm{iii})} = \begin{pmatrix} -\sin(\alpha)\sin(\alpha_{12}) e^{-\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & \sin(\alpha)\cos(\alpha_{12}) e^{-\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & -\cos(\alpha) - e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & 0 \\ \cos(\alpha_{12})\sin(\alpha_4) e^{\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} - 6\alpha_5 - 12\alpha_7\right)} & \sin(\alpha_{12})\sin(\alpha_4) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15} - 6\alpha_5 - 12\alpha_7\right)} & 0 & -e^{-\frac{1}{2} i \left(\sqrt{6}\alpha_{15} + 4\alpha_7\right)} \cos(\alpha_4) \\ \cos(\alpha_{12})\cos(\alpha_4) \left(-e^{\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15}\right)}\right) & -\sin(\alpha_{12})\cos(\alpha_4) e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15}\right)} & 0 & -e^{i\alpha_5 - i\sqrt{\frac{3}{2}}\alpha_{15}} \sin(\alpha_4) \\ \cos(\alpha)\sin(\alpha_{12}) e^{-\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & -\cos(\alpha)\cos(\alpha_{12}) e^{-\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & -\sin(\alpha) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & 0 \end{pmatrix} \tag{C.3.20}$$

For case (iv): $\alpha_1 = -\alpha_3 - \alpha_5, \alpha_2 = 0, \alpha_4 = 0, \alpha_6 = \pi/2 - \alpha$

$$D_{(\mathrm{iv})} = \begin{pmatrix} \sin(\alpha)\left(-\sin(\alpha_{12})\sin(\alpha_8) + \cos(\alpha_{10}) e^{2i(\alpha_{11} + \alpha_9)} \cos(\alpha_{12})\cos(\alpha_8)\right) e^{-\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & \sin(\alpha)\left(\cos(\alpha_{12})\sin(\alpha_8) + \cos(\alpha_{10}) e^{2i(\alpha_{11} + \alpha_9)} \sin(\alpha_{12})\cos(\alpha_8)\right) e^{-\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} \\ \left(\sin(\alpha_{12})(-\cos(\alpha_8)) - \cos(\alpha_{10}) e^{2i(\alpha_{11} + \alpha_9)} \cos(\alpha_{12})\sin(\alpha_8)\right) e^{-\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} + 6\alpha_7 + 6\alpha_9\right)} & \left(\cos(\alpha_{12})\cos(\alpha_8) - \cos(\alpha_{10}) e^{2i(\alpha_{11} + \alpha_9)} \sin(\alpha_{12})\sin(\alpha_8)\right) e^{-\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} + 6\alpha_7 + 6\alpha_9\right)} \\ \sin(\alpha_{10})\cos(\alpha_{12}) \left(-e^{\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15}\right)}\right) & \sin(\alpha_{10})\sin(\alpha_{12}) \left(-e^{\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} + 2\sqrt{3}\alpha_{14} + \sqrt{6}\alpha_{15}\right)}\right) \\ \cos(\alpha)\left(\sin(\alpha_{12})\sin(\alpha_8) - \cos(\alpha_{10}) e^{2i(\alpha_{11} + \alpha_9)} \cos(\alpha_{12})\cos(\alpha_8)\right) e^{-\frac{1}{6} i \left(6\alpha_{11} - 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)} & \cos(\alpha)\left(\cos(\alpha_{12})\sin(\alpha_8) + \cos(\alpha_{10}) e^{2i(\alpha_{11} + \alpha_9)} \sin(\alpha_{12})\cos(\alpha_8)\right) \left(-e^{-\frac{1}{6} i \left(6\alpha_{11} + 6\alpha_{13} - 2\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6\alpha_7 + 6\alpha_9\right)}\right) \end{pmatrix}$$

$$\begin{pmatrix} \ldots & \sin(\alpha)\sin(\alpha_{10})\cos(\alpha_8) e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)} & e^{-i\sqrt{\frac{3}{2}}\alpha_{15}} \cos(\alpha) \\ \ldots & \sin(\alpha_{10})\sin(\alpha_8) \left(-e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} + 6\alpha_7 - 6\alpha_9\right)}\right) & 0 \\ \ldots & \cos(\alpha_{10}) e^{\frac{i\alpha_{15}}{\sqrt{6}} - \frac{2i\alpha_{14}}{\sqrt{3}}} & 0 \\ \ldots & \cos(\alpha)\sin(\alpha_{10})\cos(\alpha_8) \left(-e^{-\frac{1}{6} i \left(4\sqrt{3}\alpha_{14} - \sqrt{6}\alpha_{15} - 6(\alpha_7 + \alpha_9)\right)}\right) & e^{-i\sqrt{\frac{3}{2}}\alpha_{15}} \sin(\alpha) \end{pmatrix} \tag{C.3.21}$$

C.4 The Haar measure

The Haar measure provides an invariant volume for a group. We would like to find the volume associated with the different subsets of matrices $D_{(I)}$.

C.4.1 Translational invariance

We first look at an analogy of the Haar measure in one dimension. The Haar measure is left and right translational invariant. We would like to understand the concept of translational invariance.

We define

$$\langle f \rangle = \int_{-\infty}^{\infty} f(x)\rho(x)dx\,, \tag{C.4.1}$$

where $\rho(x)$ is a weight function. This integral converges for all x. We also define:

$$f_a(x) = f(x-a)\,. \tag{C.4.2}$$

f_a is the function obtained after translation of the argument of f by a. Our goal is to obtain the weight function $\rho(x)$ such that $\langle f \rangle = \langle f_a \rangle$. We want the integral to be invariant under translation.

Therefore we want:

$$\int_{-\infty}^{\infty} f(x)\rho(x)dx = \int_{-\infty}^{\infty} f(x-a)\rho(x)dx\,. \tag{C.4.3}$$

We make the change of variable $y = x - a$ on the right hand side, and $y = x$ on the left hand side and obtain:

$$\begin{aligned} &\int_{-\infty}^{\infty} f(y)\rho(y)dy = \int f(y)\rho(y+a)dx \\ \Leftrightarrow &\int_{-\infty}^{\infty} f(y)(\rho(y)-\rho(y+a))dy = 0 \\ \Leftrightarrow &\rho(y)-\rho(y+a) = 0 \Leftrightarrow \rho(y) = \rho(y+a)\,, \end{aligned} \tag{C.4.4}$$

so $\rho(x) = \rho(0)$ is a constant. A function that is a constant is translation-invariant.

This was an easy example to see what a weight function is and how it permits to get a particular invariance on a function. Actually we can calculate explicitly:

$$\int_{-\infty}^{\infty} f(x)\rho(0)dx = \rho(0)(F(\infty)-F(-\infty)) = \rho(0)(F(\infty-a)-F(-\infty-a))\,. \tag{C.4.5}$$

C.4.2 Haar measure for unitary groups

Consider a unitary matrix U:

$$U = \begin{pmatrix} u_{11} & \dots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \dots & u_{nn} \end{pmatrix}. \tag{C.4.6}$$

For unitary groups, the Haar measure is determined for a general matrix as [53]:

$$dV = J\prod_i d\alpha_i\,, \tag{C.4.7}$$

where $J = \left|\det\frac{\partial(u_{11},\dots,u_{nn})}{\partial(\alpha_1,\dots\alpha_{n^2})}\right|$ .. α_i are the angles for the Euler angles parametrization. For a unitary group $U(n)$ there are n^2 independent parameters.

We can calculate explicitly the formula for $U(2)$. We consider the parametrization

$$U = e^{i\varphi/4}\begin{pmatrix} e^{i\varphi_1}\cos\theta & e^{i\varphi_2}\sin\theta \\ -e^{-i\varphi_2}\sin\theta & e^{-i\varphi_1}\cos\theta \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}. \tag{C.4.8}$$

With a Jacobian matrix is

$$\frac{\partial(u_{11},u_{12},u_{21},u_{22})}{\partial(\varphi,\varphi_1,\varphi_2,\theta)} = \begin{pmatrix} \frac{1}{4}ie^{\frac{i\varphi}{4}+i\varphi_1}\cos\theta & ie^{\frac{i\varphi}{4}+i\varphi_1}\cos\theta & 0 & -e^{\frac{i\varphi}{4}+i\varphi_1}\sin\theta \\ \frac{1}{4}ie^{\frac{i\varphi}{4}+i\varphi_2}\sin\theta & 0 & ie^{\frac{i\varphi}{4}+i\varphi_2}\sin\theta & e^{\frac{i\varphi}{4}+i\varphi_2}\cos\theta \\ -\frac{1}{4}ie^{\frac{i\varphi}{4}-i\varphi_2}\sin\theta & 0 & ie^{\frac{ivar\phi}{4}-i\varphi_2}\sin\theta & -e^{\frac{i\varphi}{4}-i\varphi_2}\cos\theta \\ \frac{1}{4}ie^{\frac{i\varphi}{4}-i\phi_1}\cos\theta & -ie^{\frac{i\varphi}{4}-i\phi_1}\cos\theta & 0 & -e^{\frac{i\varphi}{4}-i\varphi_1}\sin\theta \end{pmatrix} . \tag{C.4.9}$$

Therefore $J = |\frac{1}{2}\sin 2\theta|$, giving a Haar measure for $SU(2)$

$$dV_{U(2)} = |\frac{1}{2}\sin 2\theta| d\varphi d\varphi_1 d\varphi_2 d\theta \, . \tag{C.4.10}$$

A similar method can be used to calculate the Haar measure for $SU(3)$ and $SU(4)$ using Mathematica, or following the methods in [51, 52, 48]:

$$dV_{SU(4)} = \cos(\alpha_4)^3\cos(\alpha_6)\cos(\alpha_{10})\sin(2\alpha_2)\sin(\alpha_4)\sin(\alpha_6)^5\sin(2\alpha_8)\sin(\alpha_{10})^3\sin(2\alpha_{12}) d\alpha_{15}\ldots d\alpha_1 \, . \tag{C.4.11}$$

C.5 The single qubit case

We consider the quantum state

$$|\psi\rangle = \cos\alpha\,|\uparrow\rangle + \sin\alpha\,|\downarrow\rangle \ , \tag{C.5.1}$$

in the 2-dimensional Hilbert space $\mathcal{H}$. There is no measurement apparatus, but we wish to use the same method as in the 2-qubit case to reproduce a collapse of the wave function in either $|\uparrow\rangle$ (case (i)) or $|\downarrow\rangle$ (case (ii)). This model is not trying to reproduce a measurement process by a change of TPS in the global Hilbert space but to see that local reorientation of the measurement apparatus are allowed by unitary transformations. This would explain some of the challenges faced in the 2-qubit state.

The density matrix of $|\psi\rangle$ is:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix} . \tag{C.5.2}$$

The entropy of this system is 0 as it is a pure state After a collapse, the density matrix can take two forms:

$$\rho'_{(\mathrm{i})} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \rho'_{(\mathrm{ii})} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{C.5.3}$$

Both have entropy 0. In order to diagonalize the density matrix into one of the final form in (C.5.3), we have to find matrices D such that:

$$\rho = D_{(\mathrm{i})}\rho'_{(\mathrm{i})}D^\dagger_{(\mathrm{i})} \quad \text{or} \quad \rho = D_{(\mathrm{ii})}\rho'_{(\mathrm{ii})}D^\dagger_{(\mathrm{ii})} \, , \tag{C.5.4}$$

which is equivalent to

$$\rho D_{(\mathrm{i})} = D_{(\mathrm{i})}\rho'_{(\mathrm{i})} \quad \text{and} \quad \rho D_{(\mathrm{ii})} = D_{(\mathrm{ii})}\rho'_{(\mathrm{ii})} \, . \tag{C.5.5}$$

A 2×2 unitary matrix can be parametrized using Euler angles parametrization:

$$D = e^{i\varphi/4}\begin{pmatrix} e^{i\varphi_1}\cos\theta & e^{i\varphi_2}\sin\theta \\ -e^{-i\varphi_2}\sin\theta & e^{-i\varphi_1}\cos\theta \end{pmatrix} . \tag{C.5.6}$$

In case (i), using (C.5.4), we have

$$\begin{pmatrix} \cos^2\theta & -e^{i(\varphi_1+\varphi_2)}\cos\theta\sin\theta \\ -e^{-i(\varphi_1+\varphi_2)}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix} , \tag{C.5.7}$$

therefore,

$$\begin{cases} \theta = -\alpha \\ \varphi_1+\varphi_2 = 0 \end{cases} . \tag{C.5.8}$$

The diagonalizing matrix is written:

$$D_{(\mathrm{i})} = e^{i\varphi/4} \begin{pmatrix} e^{i\varphi_1}\cos\alpha & -e^{-i\varphi_1}\sin\alpha \\ e^{i\varphi_1}\sin\alpha & e^{-i\varphi_1}\cos\alpha \end{pmatrix}, \tag{C.5.9}$$

In case (ii):

$$\begin{cases} \theta = -\alpha + \pi/2 \\ \varphi_1 + \varphi_2 = 0 \end{cases}, \tag{C.5.10}$$

and

$$D_{(\mathrm{ii})} = e^{i\varphi/4} \begin{pmatrix} e^{i\varphi_1}\sin\alpha & e^{-i\varphi_1}\cos\alpha \\ -e^{i\varphi_1}\cos\alpha & e^{-i\varphi_1}\sin\alpha \end{pmatrix}. \tag{C.5.11}$$

Matrix $D_{(\mathrm{ii})}$ is obtain by doing a rotation of α to $\alpha + \pi/2$ in matrix $D_{(\mathrm{i})}$.

Also,

$$\begin{aligned} D_{(\mathrm{i})} &= -i\sigma_y D_{(\mathrm{ii})}\,, \\ D_{(\mathrm{ii})} &= i\sigma_y D_{(\mathrm{i})}\,. \end{aligned} \tag{C.5.12}$$

where σ_y is a Pauli matrix

Therefore, the two matrices are related by a unitary transformation. Even thought they bring the initial density matrix to a different outcome, we could interpret them are barely reorienting the system. In the 2-qubit system this is what is happening among the four kinds of matrices: some of them contain a reorientation of the measurement apparatus but actually give the same outcome. Those matrices have to be found in order to recover a proper notion of probability from the matrices.

Appendix D

Mathematica notebooks

Diagonalization conditions.

```
    Clear["Global`*"]
(*Density matrix for |\[Phi]> = cos\[Alpha]|00>+sin\[Alpha]|11>+*)
\[Rho] = {{Cos[\[Alpha]]^2, 0, 0, Cos[\[Alpha]]*Sin[\[Alpha]]}, {0, 0,
     0, 0}, {0, 0, 0, 0}, {Cos[\[Alpha]]*Sin[\[Alpha]], 0, 0,
    Sin[\[Alpha]]^2}};
Eigenvalues[\[Rho]];

(*Density matrices after diagonalization *)
\[Rho]1 = {{1, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}};
\[Rho]2 = {{0, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}};
\[Rho]3 = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 0}};
\[Rho]4 = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 1}};

(*D matrix*)
Dm = {{a, b, c, d}, {e, f, g, h}, {i, j, k, l}, {m, n, o, p}};

(* Calculations *)
A = Dm . \[Rho]1;
B = \[Rho] . Dm ;
Solve[Thread[Flatten /@ (A == B)], {a, b, c, d, e, f, g, h, i, j, k,
   l, m, n, o, p}] // Simplify
```

Inverse and conjugate transpose of D matrices.

```
    ClearAll["Global`*"];

D1 = {{a, -n*Tan[\[Alpha]], -o*Tan[\[Alpha]], -p*Tan[\[Alpha]]}, {0,
   f, g, h}, {0, j, k, l}, {a*Tan[\[Alpha]], n, o, p}}
D2 = {{-m, n*Cot[\[Alpha]], -o*Tan[\[Alpha]], -p*Tan[\[Alpha]]}, {e,
    0, g, h}, {i, 0, k, l}, {-a*Cot[\[Alpha]], n, o, p}};
D3 = {{a, -n*Tan[\[Alpha]], o*Cot[\[Alpha]], -p*Tan[\[Alpha]]}, {e, f,
     0, h}, {i, j, 0, l}, {-a*Cot[\[Alpha]], n, o, p}};
D4 = {{a, -n*Tan[\[Alpha]], -o*Tan[\[Alpha]], p*Cot[\[Alpha]]}, {e, f,
     g, 0}, {i, j, k, 0}, {-a*Cot[\[Alpha]], n, o, p}};
\[Rho] = {{Cos[\[Alpha]]^2, 0, 0, Sin[\[Alpha]]*Cos[\[Alpha]]}, {0, 0,
     0, 0}, {0, 0, 0, 0}, {Sin[\[Alpha]]*Cos[\[Alpha]], 0, 0,
    Sin[\[Alpha]]^2}};

MatrixForm[Simplify[Inverse[D1]]]
MatrixForm[Simplify[ConjugateTranspose[D1]]]
```

Parametrization for $SU(4)$.

```
    Clear["Global`*"]
(*PARAMETRIZATION U(4)*)

(*Generators of U(4) needed for the parametrization*)
lambda2 = {{0, -I, 0, 0}, {I, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}};
lambda3 = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}};
lambda5 = {{0, 0, -I, 0}, {0, 0, 0, 0}, {I, 0, 0, 0}, {0, 0, 0, 0}};
lambda8 = (1/Sqrt[3]) {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -2, 0}, {0,
     0, 0, 0}};
lambda10 = {{0, 0, 0, -I}, {0, 0, 0, 0}, {0, 0, 0, 0}, {I, 0, 0, 0}};
lambda15 = (1/Sqrt[6]) {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0,
     0, 0, -3}};

(*Density matrix for |\[Psi]> = Cos(\[Alpha])|00>+Sin(\[Alpha])|11>*)
Rho = {{Cos[\[Alpha]]^2, 0, 0, Cos[\[Alpha]]*Sin[\[Alpha]]}, {0, 0, 0,
     0}, {0, 0, 0, 0}, {Cos[\[Alpha]]*Sin[\[Alpha]], 0, 0,
    Sin[\[Alpha]]^2}};

(*4x4 Unitary matrix*)
Uni = MatrixExp[(I \[Alpha]1) lambda3] .
   MatrixExp[(I \[Alpha]2) lambda2] .
   MatrixExp[(I \[Alpha]3) lambda3] .
   MatrixExp[(I \[Alpha]4) lambda5] .
   MatrixExp[(I \[Alpha]5) lambda3] .
   MatrixExp[(I \[Alpha]6) lambda10] .
   MatrixExp[(I \[Alpha]7) lambda3] .
   MatrixExp[(I \[Alpha]8) lambda2] .
   MatrixExp[(I \[Alpha]9) lambda3] .
   MatrixExp[(I \[Alpha]10) lambda5] .
   MatrixExp[(I \[Alpha]11) lambda3] .
   MatrixExp[(I \[Alpha]12) lambda2] .
   MatrixExp[(I \[Alpha]13) lambda3] .
   MatrixExp[(I \[Alpha]14) lambda8] .
   MatrixExp[(I \[Alpha]15) lambda15];

(*Show the matrix*)
Uni // MatrixForm
```

Diagonalization condition on unitary matrices.

```
    Clear["Global`*"]
(*WRITE THE PARAMETRIZED MATRICES IN EACH CASES*)
(*Generators of U(4) needed for the parametrization*)
lambda2 = {{0, -I, 0, 0}, {I, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}};
lambda3 = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}};
lambda5 = {{0, 0, -I, 0}, {0, 0, 0, 0}, {I, 0, 0, 0}, {0, 0, 0, 0}};
lambda8 = (1/Sqrt[3]) {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, -2, 0}, {0,
      0, 0, 0}};
lambda10 = {{0, 0, 0, -I}, {0, 0, 0, 0}, {0, 0, 0, 0}, {I, 0, 0, 0}};
lambda15 = (1/Sqrt[6]) {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0,
      0, 0, -3}};
(*Density matrix for |\[Psi]> = Cos(\[Alpha])|00>+Sin(\[Alpha])|11>*)
Rho = {{Cos[\[Alpha]]^2, 0, 0, Cos[\[Alpha]]*Sin[\[Alpha]]}, {0, 0, 0,
      0}, {0, 0, 0, 0}, {Cos[\[Alpha]]*Sin[\[Alpha]], 0, 0,
     Sin[\[Alpha]]^2}};

(*Chose the case 1= |\[Psi]> -> |00>, 2= |\[Psi]> -> |01>, 1= \
|\[Psi]> -> |10>, 1= |\[Psi]> -> |11>*)
Case = 1
If[Case ==
   1, {\[Alpha]2 = Pi/2, \[Alpha]12 = Pi/2, \[Alpha]6 =
    Pi/2, \[Alpha]10 = -(Pi/
        4 - \[Alpha]), \[Alpha]8 = -\[Alpha], \[Alpha]1 = \[Alpha]3 + \
\[Alpha]5 + 2 \[Alpha]7}];
(*\[Alpha]6=Pi/2,\[Alpha]12=Pi/2,\[Alpha]2=Pi/2,\[Alpha]1=\[Alpha]3+\
\[Alpha]5+2 \[Alpha]7,*)
If[Case ==
   2, {\[Alpha]6 = Pi/2, \[Alpha]12 = 0, \[Alpha]2 = Pi/2, \[Alpha]8 =
     Pi - \[Alpha], \[Alpha]1 = \[Alpha]3 + \[Alpha]5 +
     2 \[Alpha]7}];
If[Case ==
   3, {\[Alpha]6 = Pi/2, \[Alpha]10 = Pi/2, \[Alpha]2 =
    Pi/2, \[Alpha]1 = \[Alpha]3 + \[Alpha]5 + 2 \[Alpha]7, \[Alpha]8 =
     3*Pi/2 - \[Alpha]}];
If[Case ==
   4, {\[Alpha]2 = 0, \[Alpha]4 =
    0, \[Alpha]1 = -\[Alpha]3 - \[Alpha]5, \[Alpha]6 =
    3*Pi/2 - \[Alpha]}];

(*Compute the Unitary Matrix in the chosen case*)
Uni = MatrixExp[(I \[Alpha]1) lambda3] .
   MatrixExp[(I \[Alpha]2) lambda2] .
   MatrixExp[(I \[Alpha]3) lambda3] .
   MatrixExp[(I \[Alpha]4) lambda5] .
   MatrixExp[(I \[Alpha]5) lambda3] .
   MatrixExp[(I \[Alpha]6) lambda10] .
   MatrixExp[(I \[Alpha]7) lambda3] .
   MatrixExp[(I \[Alpha]8) lambda2] .
   MatrixExp[(I \[Alpha]9) lambda3] .
   MatrixExp[(I \[Alpha]10) lambda5] .
   MatrixExp[(I \[Alpha]11) lambda3] .
   MatrixExp[(I \[Alpha]12) lambda2] .
   MatrixExp[(I \[Alpha]13) lambda3] .
   MatrixExp[(I \[Alpha]14) lambda8] .
   MatrixExp[(I \[Alpha]15) lambda15];

(*Show the matrix*)
Uni // MatrixForm // Simplify
(*Check if it diagonalizes into the chose case*)
MatrixForm[
   Simplify[
    ConjugateTranspose[Uni] . Rho .
     Uni, {\[Alpha]9 \[Element] Reals, \[Alpha]10 \[Element]
       Reals, \[Alpha]11 \[Element] Reals, \[Alpha]12 \[Element]
       Reals, \[Alpha]13 \[Element] Reals, \[Alpha]14 \[Element]
       Reals, \[Alpha]15 \[Element] Reals, \[Alpha]1 \[Element]
       Reals, \[Alpha]2 \[Element] Reals, \[Alpha]3 \[Element]
       Reals, \[Alpha]4 \[Element] Reals, \[Alpha]5 \[Element]
       Reals, \[Alpha]6 \[Element] Reals, \[Alpha]7 \[Element]
       Reals, \[Alpha]8 \[Element] Reals, \[Alpha] \[Element]
       Reals}]] // Simplify // MatrixForm
```

Entanglement entropy for a 2-qubit state with real coefficients.

```
Clear["Global`*"]
(*ENTROPY FOR 2 QUBIT STATE PARAMETRIZED WITH ANGLES*)

(*Entropy formular for |\[Psi]> =Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\
\[Theta]]|00>+Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]|01>+Cos[\
\[Alpha]]*Cos[\[Beta]]|10>+Sin[\[Alpha]]|11>*)
S[\[Alpha]_, \[Beta]_, \[Theta]_]:= -(1 +
       Sqrt[1 -
         4 (Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\[Theta]]*Sin[\[Alpha]] -
            Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]*Cos[\[Alpha]]*
             Cos[\[Beta]])^2])/2*
   Log[(1 +
       Sqrt[1 -
         4 (Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\[Theta]]*Sin[\[Alpha]] -
            Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]*Cos[\[Alpha]]*
             Cos[\[Beta]])^2])/
     2] - (1 -
      Sqrt[1 -
        4 (Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\[Theta]]*Sin[\[Alpha]] -
           Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]*Cos[\[Alpha]]*
            Cos[\[Beta]])^2])/2*
   Log[(1 -
       Sqrt[1 -
         4 (Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\[Theta]]*Sin[\[Alpha]] -
            Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]*Cos[\[Alpha]]*
             Cos[\[Beta]])^2])/2]

(*Get the value depending on the coefficients *)
Manipulate[{{N[
    S[\[Alpha], \[Beta], \[Theta]]/Log[2]]}, {Cos[\[Alpha]]*
     Sin[\[Beta]]*Cos[\[Theta]] Text["|00>"] +
    Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]] Text["|01>"] +
    Cos[\[Alpha]]*Cos[\[Beta]] Text["|10>"] +
    Sin[\[Alpha]] Text[
      "|11>"]}, {"Indeterminates means S=0"}}, {\[Theta], 0, Pi,
  Pi/8}, {\[Alpha], 0, Pi, Pi/8}, {\[Beta], 0, Pi, Pi/8}]
(*Plot3D*)
Manipulate[
 Plot3D[S[\[Alpha], \[Beta], \[Theta]]/Log[2], {\[Alpha],
   0, \[Pi]}, {\[Beta], 0, \[Pi]}, AxesLabel -> Automatic,
  Ticks -> {{0, \[Pi]/4, \[Pi]/2,
     3 \[Pi]/4, \[Pi]}, {0, \[Pi]/4, \[Pi]/2, 3 \[Pi]/4, \[Pi]}, {0,
     0.5, 1}}],
 {\[Theta], 0, \[Pi]}]

(*ContourPlot*)
Manipulate[
 Grid[{{Row[{"\[Theta]=", \[Theta]}]}, {ContourPlot[
     S[\[Alpha], \[Beta], \[Theta]]/Log[2], {\[Alpha],
      0, \[Pi]}, {\[Beta], 0, \[Pi]}, PlotLegends -> Automatic,
     FrameLabel -> Automatic]}}, Spacings -> {1, 1}, Frame -> All],
 {\[Theta], 0, \[Pi], Pi/8}]
```

Make a video of the evolution of the entanglment entropy for a two-qubits state with varying coefficients.

```
    Clear["Global`*"]
(*VIDEO FOR ENTROPY FOR 2 QUBIT STATE PARAMETRIZED WITH ANGLES*)

(*Entropy formular for |\[Psi]> =Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\
\[Theta]]|00>+Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]|01>+Cos[\
\[Alpha]]*Cos[\[Beta]]|10>+Sin[\[Alpha]]|11>*)
S[\[Alpha]_, \[Beta]_, \[Theta]_]:= -(1 +
       Sqrt[1 -
         4 (Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\[Theta]]*Sin[\[Alpha]] -
            Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]*Cos[\[Alpha]]*
             Cos[\[Beta]])^2])/2*
   Log[(1 +
       Sqrt[1 -
         4 (Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\[Theta]]*Sin[\[Alpha]] -
            Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]*Cos[\[Alpha]]*
             Cos[\[Beta]])^2])/
     2] - (1 -
      Sqrt[1 -
        4 (Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\[Theta]]*Sin[\[Alpha]] -
           Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]*Cos[\[Alpha]]*
            Cos[\[Beta]])^2])/2*
   Log[(1 -
       Sqrt[1 -
         4 (Cos[\[Alpha]]*Sin[\[Beta]]*Cos[\[Theta]]*Sin[\[Alpha]] -
            Cos[\[Alpha]]*Sin[\[Beta]]*Sin[\[Theta]]*Cos[\[Alpha]]*
             Cos[\[Beta]])^2])/2]

(*Video Contour*)
  (*Without grid*)
fframe[\[Theta]_]:=
  ContourPlot[
   S[\[Alpha], \[Beta], \[Theta]]/Log[2], {\[Alpha],
    0, \[Pi]}, {\[Beta], 0, \[Pi]}, PlotLegends -> Automatic,
   FrameLabel -> Automatic];
  (*With grid*)
frame[\[Theta]_]:=
 Grid[{{Row[{"\[Theta]=", \[Theta]}]}, {ContourPlot[
      S[\[Alpha], \[Beta], \[Theta]]/Log[2], {\[Alpha],
       0, \[Pi]}, {\[Beta], 0, \[Pi]}, PlotLegends -> Automatic,
      FrameLabel -> Automatic]}}, Spacings -> {1, 1}, Frame -> All]

(*Video 3D*)
frame3D[\[Theta]_]:=
 Grid[{{Row[{"\[Theta]=", \[Theta]}]}, {Plot3D[
      S[\[Alpha], \[Beta], \[Theta]]/Log[2], {\[Alpha],
       0, \[Pi]}, {\[Beta], 0, \[Pi]}, AxesLabel -> Automatic,
      Ticks -> {{0, \[Pi]/4, \[Pi]/2,
         3 \[Pi]/4, \[Pi]}, {0, \[Pi]/4, \[Pi]/2,
         3 \[Pi]/4, \[Pi]}, {0, 0.5, 1}}]}}, Spacings -> {1, 1},
  Frame -> All]

(*Generate all the pictures*)
frames = ParallelTable[frame3D[\[Theta]], {\[Theta], 0, 2*Pi, Pi/256}];
(*Make the video*)
Export["Entropyb.avi", frames, FrameRate -> 24]


(*Make an image with three plots*)
tableEntropy = TableForm[{{frame[0], frame[Pi/4], frame[Pi/2]}}]
Export["entropy.png", tableEntropy]
(*Make some pictures*)
(*Export["entropy0.png",frame[0]]
Export["entropyPi/2.png",frame[Pi/2]]
Export["entropyPi.png",frame[Pi]]*)
```

Bibliography

[1] E. Schrödinger. An undulatory theory of the mechanics of atoms and molecules. *Phys. Rev.*, 28:1049–1070, Dec 1926.

[2] W. Heisenberg. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift fur Physik*, 43(3-4):172–198, March 1927.

[3] Paolo Zanardi, Daniel A. Lidar, and Seth Lloyd. Quantum tensor product structures are observable induced. *Physical Review Letters*, 92(6), feb 2004.

[4] Jordan S. Cotler, Geoffrey R. Penington, and Daniel H. Ranard. Locality from the spectrum. *Communications in Mathematical Physics*, 368(3):1267–1296, feb 2019.

[5] Sean M. Carroll and Ashmeet Singh. Quantum mereology: Factorizing Hilbert space into subsystems with quasiclassical dynamics. *Phys. Rev. A*, 103(2):022213, 2021.

[6] J. P. Gordon. Quantum effects in communications systems. *Proceedings of the IRE*, 50(9):1898–1908, 1962.

[7] K. Przibram (Foreword) Albert Einstein (Author), Martin J. Klein (Introduction). *Letters on Wave Mechanics: Correspondence with H. A. Lorentz, Max Planck, and Erwin Schrödinger*. Philosophical Library Open Road, 2011.

[8] Tim Maudlin. Three measurement problems. *Topoi*, 14(1):7–15, 1995.

[9] Hugh Everett. "Relative State" Formulation of Quantum Mechanics. *Reviews of Modern Physics*, 29(3):454–462, July 1957.

[10] David Bohm. A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I. *Physical Review*, 85(2):166–179, January 1952.

[11] David Bohm. A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II. *Physical Review*, 85(2):180–193, January 1952.

[12] C. E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27(3):379–423, 1948.

[13] Charles H. Bennett and Gilles Brassard. Quantum cryptography: Public key distribution and coin tossing. *Theoretical Computer Science*, 560:7–11, dec 2014.

[14] Ekert. Quantum cryptography based on bell's theorem. *Physical review letters*, 67 6:661–663, 1991.

[15] Peter W. Shor. Scheme for reducing decoherence in quantum computer memory. , 52(4):R2493–R2496, October 1995.

[16] P. A. M. Dirac. A new notation for quantum mechanics. *Mathematical Proceedings of the Cambridge Philosophical Society*, 35(3):416–418, 1939.

[17] D. Hilbert, J. von Neumann, and L. Nordheim. Über die grundlagen der quantenmechanik. *Mathematische Annalen*, 98:1–30, 1928.

[18] J. Audretsch. *Entangled Systems: New Directions in Quantum Physics*. John Wiley & Sons, Ltd, 2007.

[19] W. Gasser. J. v. neumann, mathematische grundlagen der quantenmechanik. (die grundlehren der math. wissenschaften in einzeldarstellungen, band 38). viii + 262 s. m. 4 abb. berlin/heidelberg/new york 1968. springer-verlag. preis geb. dm 28, - . *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 50(6):437–438, 1970.

[20] A. Einstein, B. Podolsky, and N. Rosen. Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Physical Review*, 47(10):777–780, May 1935.

[21] Andrea Di Biagio, Richard Howl, Časlav Brukner, Carlo Rovelli, and Marios Christodoulou. Relativistic locality can imply subsystem locality, 2023.

[22] W. H. Zurek. Pointer Basis of Quantum Apparatus: Into What Mixture Does the Wave Packet Collapse? *Phys. Rev. D*, 24:1516–1525, 1981.

[23] Carlos Alexandre Brasil and Leonardo Andreta de Castro. Understanding the pointer states. *European Journal of Physics*, 36(6):065024, sep 2015.

[24] Wojciech H. Zurek. Preferred observables, predictability, classicality, and the environment-induced decoherence, 1994.

[25] M. Schlosshauer. *Decoherence and the Quantum-to-Classical Transition.* Springer Berlin, Heidelberg, 2007.

[26] N. Bohr. Can Quantum-Mechanical Description of Physical Reality be Considered Complete? *Physical Review*, 48(8):696–702, October 1935.

[27] J. S. Bell. On the einstein podolsky rosen paradox. *Physics Physique Fizika*, 1:195–200, Nov 1964.

[28] Stuart J. Freedman and John F. Clauser. Experimental test of local hidden-variable theories. *Phys. Rev. Lett.*, 28:938–941, Apr 1972.

[29] Alain Aspect. Proposed experiment to test the nonseparability of quantum mechanics. *Phys. Rev. D*, 14:1944–1951, Oct 1976.

[30] Alain Aspect, Jean Dalibard, and Gérard Roger. Experimental test of bell's inequalities using time-varying analyzers. *Phys. Rev. Lett.*, 49:1804–1807, Dec 1982.

[31] Kusch, Valentini, Gefter, Spekkens, Zeilinger, Fuchs Rovelli, Stadler, and Kastner. Lecture series on scientific realism, 2018.

[32] H. D. Zeh. On the interpretation of measurement in quantum theory. *Foundations of Physics*, 1(1):69–76, March 1970.

[33] Carlo Rovelli. Relational quantum mechanics. *International Journal of Theoretical Physics*, 35(8):1637–1678, aug 1996.

[34] Christopher G. Timpson. Quantum bayesianism: A study, 2008.

[35] Maximilian Schlosshauer, Johannes Kofler, and Anton Zeilinger. A snapshot of foundational attitudes toward quantum mechanics. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 44(3):222–230, aug 2013.

[36] Max Tegmark. The interpretation of quantum mechanics: Many worlds or many words? *Fortschritte der Physik*, 46(6-8):855–862, nov 1998.

[37] Max Born. Zur quantenmechanik der stoßvorgänge. *Zeitschrift für Physik*, 37:863–867, 1926.

[38] Charles P. Pauli, Wolfgang Enz and Karl V. Meyenn. *Writings on Physics and Philosophy.* 1994.

[39] G. C. Ghirardi, A. Rimini, and T. Weber. Unified dynamics for microscopic and macroscopic systems. *Phys. Rev. D*, 34:470–491, Jul 1986.

[40] Gian Carlo Ghirardi, Philip Pearle, and Alberto Rimini. Markov processes in hilbert space and continuous spontaneous localization of systems of identical particles. *Phys. Rev. A*, 42:78–89, Jul 1990.

[41] L. Diósi. A universal master equation for the gravitational violation of quantum mechanics. *Physics Letters A*, 120(8):377–381, 1987.

[42] Bryce S. DeWitt. Quantum mechanics and reality. *Physics Today*, 23(9):30–35, 09 1970.

[43] Sean M. Carroll and Charles T. Sebens. Many worlds, the born rule, and self-locating uncertainty. In Daniele C. Struppa and Jeffrey M. Tollaksen, editors, *Quantum Theory: A Two-Time Success Story*, pages 157–169, Milano, 2014. Springer Milan.

[44] Carlton M. Caves, Christopher A. Fuchs, and Rüdiger Schack. Quantum probabilities as bayesian probabilities. *Physical Review A*, 65(2), jan 2002.

[45] Christopher A. Fuchs. Qbism, the perimeter of quantum bayesianism, 2010.

[46] Robert B. Griffiths. Consistent histories and the interpretation of quantum mechanics. *Journal of Statistical Physics*, 36(1-2):219–272, July 1984.

[47] John G. Cramer. The transactional interpretation of quantum mechanics. *Rev. Mod. Phys.*, 58:647–687, Jul 1986.

[48] Todd Tilma, Mark Byrd, and E C G Sudarshan. A parametrization of bipartite systems based on $su(4)$ euler angles. *Journal of Physics A: Mathematical and General*, 35(48):10445–10465, nov 2002.

[49] R. Landauer. Irreversibility and heat generation in the computing process. *IBM Journal of Research and Development*, 5(3):183–191, 1961.

[50] Erhard Schmidt. Zur theorie der linearen und nichtlinearen integralgleichungen. i. teil: Entwicklung willkürlicher funktionen nach systemen vorgeschriebener. *Mathematische Annalen*, 63:433–476, 1907.

[51] Mark Byrd. The geometry of su(3), 1997.

[52] Mark Byrd. Differential geometry on SU(3) with applications to three state systems. *Journal of Mathematical Physics*, 39(11):6125–6136, nov 1998.

[53] Christoph Spengler, Marcus Huber, and Beatrix C. Hiesmayr. Composite parameterization and haar measure for all unitary and special unitary groups. *Journal of Mathematical Physics*, 53(1):013501, jan 2012.