KTH Engineering Sciences

Master Thesis

Covariant Prescription of Holographic Entanglement Entropy in AdS_3 and BTZ Black Hole

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Stockholm, Sweden 2015

Typeset in $\LaTeX$

TRITA-FYS-2015:40
ISSN 0280-316X
ISRN KTH/FYS/--15:40--SE

Printed in Sweden by Universitetsservice US AB, Stockholm June 2015

Abstract

In this thesis, using the replica trick, I compute the time-dependent entanglement entropy for three different conformal field theories (CFT): CFT on the real line at zero temperature, CFT on the circle at zero temperature, and the CFT on the real line at finite temperature. I compare the results with holographic covariant entanglement entropy proposed by Hubeny, Rangamani and Takanayagi in [1], that uses geodesics in: AdS_3 in Poincaré's and global coordinates, and he BTZ black hole respectively. Both methods match perfectly and I present the details and subtleties of the computations.

Preface

This thesis is the result of my Master of Science degree project done at the Department of Theoretical Physics at the Royal Institute of Technology (KTH) during the spring of 2015. I thank my supervisor Pawel Caputa for introducing me to an interesting topic, for the stimulating discussions and feedback. I also want to thank to my supervisor Edwin Langmann for giving me the opportunity to write the Master of Science Thesis and also for giving me feedback.

Stockholm, June 2015
Mario Benites

Contents

Chapter 1

Introduction

For decades, relativistic quantum field theory (QFT) is a framework to describe the observed behaviour and properties of elementary particles. QFT describes the interactions between elementary particles. For example, we have quantum electrodynamics (QED) which studies the interactions between electrically charged particles by means of exchange of photons, and quantum chromodynamics [2] which describes the interactions between quarks and gluons which make up the hadrons, i.e(proton, neutron, pion). These theories work well only when the gravitational interactions are sufficiently small that we can neglect the gravitational effects.

On the other hand we have General Relativity, which so far has been giving great insights about orbits of planets, the evolution of the galaxies, the Big Bang, black holes, etc. It unifies the description of gravity as a geometric property of space and time. However, the problem is that this theory has not been related with quantum mechanics yet. So far it has been very difficult to incorporate GR into the QFT. The most prominent candidate seems to be string theory.

String theory was initially proposed to explain the observed relationship between mass and spin of hadrons. Nevertheless, it turned out to be a theory of quantum gravity. The main idea of string theory is replacing the concept of point-like particles with one dimensional objects called strings. This means that the charge, mass and other properties are determined by the vibrational state. In this sense the gravitons would correspond to a closed string in a low energy vibrational state, which explains why gravitation is the weakest of the four interactions. A theory of quantum gravity could have been considered in the QFT by inserting the gravitational effects, but that results in a non-renormalizable theory, therefore it cannot be used to make any physical predictions.

One of the big results that came out of string theory is the Holographic Principle [3],[4]. It claims that the description of a volume of space can be encoded on a boundary of the region. This arose after the microscopic derivation of the Bekenstein-Hawking entropy S_{BH} for BPS Holes [5] [6] that is given by:

$$\boxed{S_A = \frac{A}{4G_N}} \tag{1.1}$$

Where A is the area of the event horizon and G_N is the gravitational constant. This is a relation between gravitational entropy and the degeneration of quantum field theory as its microscopic description. It also means that the whole information corresponding of every object that have fallen into the black hole is distributed over the area of the event horizon. This motivated Maldacena to propose the AdS/CFT correspondence [7]. This duality claims that a (d+1)-dimensional CFT (CFT_{d+1}) is equivalent to String theory on a (d+2)-dimensional anti-de Sitter space (AdS_{d+2}). It is expected that the CFT_{d+1} lives on the boundary of the AdS_{d+2} space. An example of this is the equivalence between type IIB string theory [8] compactified on $AdS_5 \times S^5$, and 4-dimensional $N = 4$ super-symmetric Yang Mills theory. Where the S^5 is a 5-dimensional sphere.

This duality has not been proven yet, but it has plenty of applications in nuclear physics, condensed matter theory and high energy physics. Although it has numerous evidences that this duality works, it is still unknown which region of AdS is responsible for the particular information in the dual CFT. Recently it is believed that in order to make progress in this specific problem, we need to formulate and study the holography in terms of a universal observable. The best candidate so far seems to be entanglement entropy.

Von Neumann entanglement entropy is the main subject of this thesis. This type of entropy is the generalized form of Gibbs entropy that measures how quantum a given wave function is. To calculate Von Neumann entropy the system is divided into two subsystems A and B. This type of entropy relies on calculating the reduced density matrix ρ_A for the subsystem A, which is obtained by tracing over the subsystem B of the total density matrix $\rho = |\Psi\rangle \langle\Psi|$.

I will perform several computations of entanglement entropies using two different methods in the AdS/CFT. The first method will be done in the CFT by using the replica trick which consists in making n copies of the system [9], [10], [11]. The local fields in the CFT will connect each copy one another. This is useful to obtain the trace $Tr(\rho_A)^n$ which is necessary for the Von Neumann entropy computation. I will use the replica trick for a general 2D CFT on a real line and on a circle as well as a CFT on a real line at finite temperature.

In AdS/CFT there exist a formula due to Ryu-Takayanagi for holographic entanglement entropy [12]. The formula resembles (1.1), but the area A is replaced by the area of the minimal surface that ends on the entangled region at the boundary. Thus, the second method consists in determining the geodesics in a specific metric of space-time. The metrics that I will consider for the holographic method are: AdS_3 in Poincaré's and global coordinates, and the BTZ black hole [13]. I will use these metrics for the non-static case as a main goal. Then, I will compare the results with those from the CFT that should be equal due to AdS/CFT.

It is really important to study the time-dependent version (covariant prescription) of entanglement entropy, because it can give us a deeper understanding about holography and quantum gravity since in QG space and time should be treated on equal footing. Moreover, from the perspective of many body systems out of equilibrium it is also desirable to have a precise notion of time-dependent degrees of freedom, which entanglement entropy is a good measure.

Chapter 2

Background Material

This chapter reviews the definition and properties of Von Neumann entanglement entropy and its basic properties discussed. I begin explaining in section (2.1) of how the quantum states $|\Psi\rangle$ are related to the density matrix ρ. This is considered in the definition of Von Neumann entropy, which requires dividing the system into two subsystems A and B. In general, this type of entropy is computed only for one of the subsystems, which in this case I will take subsystem A.

Section (2.2) reviews two computational methods of entanglement entropy. The first method is the Replica trick, which consists in replicating the World-sheet of a system and gluing them each one another by using the local operators to compute Von Neumann entropy. The second method consists in using the AdS/CFT duality to compute entanglement entropy by using an area law proposed in [12] ,[1]. Where the area in this formula corresponds to the extremized surface that ends on the entangled region at the boundary.

2.1 Entanglement Entropy and Properties

Entanglement entropy is an important tool that quantifies entanglement. In this thesis I study the case of a quantum mechanical system with many degrees of freedom. Its definition relies on dividing the system into two subsystems A and B at zero temperature. This implies that the Hilbert space of the total system can be written as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces of subsystems A and B respectively. In quantum mechanics the density matrix can be obtained from the pure state $|\Psi\rangle$ by the following:

$$\rho_{tot} = |\Psi\rangle \langle\Psi| \tag{2.1}$$

If an observer is only accessible to subsystem A, then the observer will feel as if the total system is described by the reduced density matrix ρ_A, which is:

$$\rho_A = tr_B \rho_{tot} \tag{2.2}$$

Where the trace is taken over the states of subsystem B. Then, we define the entropy of subsystem A as the Von Neumann entropy which is related to the reduced density matrix ρ_A. This is:

$$S_A = -tr(\rho_A \log(\rho_A)) \tag{2.3}$$

This is important because it measures of how closely entangled the given wave-function $|\psi >$ is. If we want to calculate the entanglement entropy $S(\beta)$ at finite temperature $T = \beta^{-1}$, we substitute $\rho_{thermal} = e^{-\beta H}$ in (2.3). By doing this, we observe that the Von Neumann entropy $S_A(\beta)$ is the thermal entropy only if A is the whole system.

Entanglement entropy satisfies the following properties:

- If the total system is in a pure quantum state, then Von Neumann entanglement entropy satisfies: S=0
- If the total system state is pure, then we obtain:

$$S_A = S_B \tag{2.4}$$

  This equality is not true when the system is in a mixed state. I will discuss this in chapter (3) and chapter (4).

- When subsystem A is divided into two submanifolds A_1 and A_2, subadditivity is satisfied:

$$S_{A_1} + S_{A_2} \geq S_A \tag{2.5}$$

  This holds with equality when the two submanifolds are uncorrelated.

- For any three subsystems A,B and C that do not intersect each other, the following strong sub-additivity inequality holds:

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C} \tag{2.6}$$

  Also, for any subsystem A and B one can have a stronger version of (2.5):

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B} \tag{2.7}$$

  This states that if subsystem A doesn't intersect with subsystem B, then the last relation reduces to the sub-additivity. These properties were previously explained in [14], [15].

2.2 Computational Methods of Entanglement Entropy

In this section I will review the two computational methods of entanglement entropy. The first method is the replica trick, which consists in replicating the world sheet of a system. These world sheets will be glued by the local fields (twist fields), which simplifies the computation of Von Neumann entropy. I will explain this method for the static case (time fixed) and when the quantum states evolve in time.

However, in the AdS/CFT the Von Neumann entropy is related to the area of the extremized surface that ends on the entangled region at the boundary. Therefore, the second method consists of extremizing the geodesic path length between two points in a metric of space and time. In general, this method is applied when the time is not fixed as a constant. Fixing the time, the extremized geodesic path length corresponds to the minimal geodesic path length which is the case studied by Ryu-Takayanagi in [12].

2.2.1 Replica Trick

Entanglement entropy for time-independent states in CFT

Before explaining the replica trick, I will make a brief summary of the CFT. CFT is a quantum field theory that is invariant under conformal transformations, which means that it locally preserves angles between any two lines. For the 2-dimensional CFT, these transformations are generated by two independent operators l_m and $\bar{l}_m$, where these generators are described by relations (A.8) and (A.9). The value of m is any integer number and these operators represent a copy of the Witt algebra.

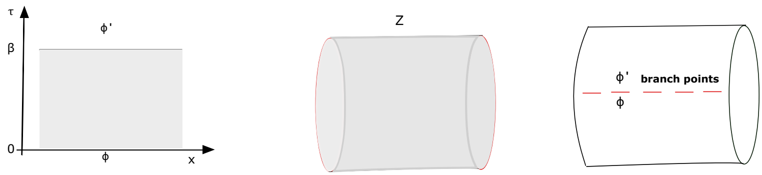
The Witt algebra of infinitesimal conformal transformations admits a central extension (A.14), this is important because this is where the central charge c of the CFT comes from. In general this central charge will appear in the computations of entanglement entropy.

The conformal transformations simplifies the computations that is not required to solve any quantum path integrals to determine the two point functions of the fields. These results of the two point functions will be used in the replica trick.

The replica trick is a powerful tool used to compute the Rényi entropy. This entropy is defined as:

$$S_A^n = \frac{1}{1-n} \log\left(tr\left(\rho_A^n\right)\right) \tag{2.8}$$

Where this is a generalization of Von Neumann entropy. Actually, we can observe that taking the limit $n \to 1$ yields the Von Neumann entropy. It is better to calculate first the Rényi entropy because it is too difficult to compute directly the Von Neumann entropy by using the quantum path integral formulation of the density matrix ρ. For the time fixed case, the quantum path integral formulation of the density matrix is:



(a) (b) (c)

Figure 2.1: Figure(a):Is the graphical form of path integral formulation of the density matrix $\rho_A(\phi, \phi')$. Figure(b): This cylinder has radius β, it represents the partition function Z which is obtained by sewing the edges along $\tau = 0$ and $\tau = \beta$ to form this cylinder shape. Figure(c): Is an illustration of how to construct the reduced density matrix $\rho_A(\phi, \phi')$ which is obtained by sewing those points together in x which are not in subsystem A

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$$\rho(\{\phi_x\}\{\phi'_{x'}\}) = Z^{-1} \int [D\phi(y,\tau)] \prod_{x'} \delta(\phi(y,0) - \phi'_{x'}) \prod_{x} \delta(\phi(y,\beta) - \phi_x) e^{-S_E} \quad (2.9)$$

Where S_E is the Euclidean action and is obtained by integrating the Lagrangian of the system between the Euclidean times $\tau = 0$ and $\tau = \beta$. This integral is obtained by considering the system as a lattice, where the lattice sites are labelled as discrete variables x. The row and column vectors of the density matrix are the fields at the boundary points: $\tau = 0, \beta$. Figure (2.1a) illustrates the graphical form of the integral version of the density matrix.

The factor Z from relation (2.9) is the partition function. According to [11] this partition function is calculated by sewing the edges together along $\tau = 0$ and $\tau = \beta$ (see figure (2.1b) for an illustration). This is done by taking $\{\phi_x\} = \{\phi'_{x'}\}$ and integrating every possible value of the fields along the boundary points.

The computation of Rényi entropy requires the reduced density matrix ρ_A. For this case, subsystem A is composed of the points of x in the disjoints intervals $(u_1, v_1), ..., (u_N, v_N)$. The reduced density matrix can be obtained by sewing together in (2.9) the points of x which are not in A. This leaves some open cuts per each interval defined in subsystem A (see figure (2.1c) for an illustration).

The Rényi entropy depends of $tr(\rho_A)^n$, where this trace is calculated by using the replica trick. This method consists in making n copies of this constructed object, and sewing them together cyclically to get a n-sheeted Riemann surface $\mathcal{R}_{n,N}$. Each surface will have the branch points defined by the interval (u_j, v_j) for every $j = 1, ..., n$. The branch points are glued cyclically by the fields (see figure (2.2)), so the trace $tr(\rho_A)^n$ is:

$$tr(\rho_A^n) = \frac{Z_n(A)}{Z^n} \quad (2.10)$$

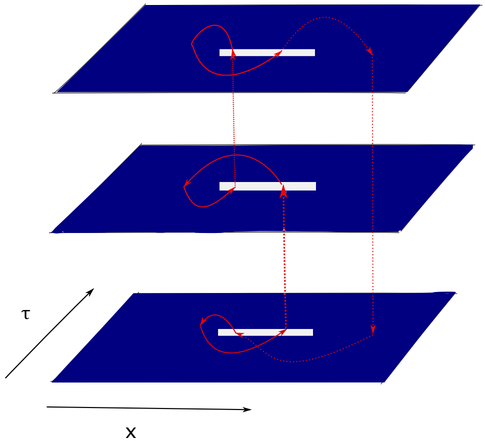


Figure 2.2: This is an example of a 3-sheeted system Riemann surface $\mathcal{R}_{3,1}$ and the arrows indicates how the points from the cut must be linked with the other points of the Riemann sheets.

where $Z_n(A)$ is the partition function over the n-sheeted surface.

The computation of entanglement entropy requires the continuous case, therefore limit $a \to 0$ must be taken. This implies that the integrals of the reduced density matrix ρ_A will be integrated over the fields on $\mathcal{R}_{n,N}$.

In this thesis I will use one interval (u_1, v_1) and the n-sheeted Riemann structure will be called as $\mathcal{R}_n$. The locality is recovered by passing from the n world-sheets to the target space. In the target space, the total Lagrangian of the replicated system will be the sum of the Lagrangian of each Riemann surface. This target space defines the local operators, which glue all the Riemann surfaces together. These local operators are known as the twist fields, which satisfy two opposite cyclic permutation symmetries: $i \to i+1$ and $i+1 \to i$. There are two type of twist fields, where in [11] are denoted as $\mathcal{T}_n$ and $\tilde{\mathcal{T}}_n$. Where the Twist field $\mathcal{T}_n$ satisfies the cyclic permutation $i \to i+1$ and $\tilde{\mathcal{T}}_n$ satisfies $i+1 \to i$.

The partition function over $\mathcal{R}_n$ will be proportional to the two-point function $\left\langle \mathcal{T}_n(u_1, 0)\tilde{\mathcal{T}}_n(v_1, 0)\right\rangle_{\mathcal{L}^n,\mathbb{C}}$. In the branch points we have conical singularities, therefore, the Rényi entropy will depend of the cut-off value a. According to [10], this implies that the trace of the reduced density matrix is:

$$tr(\rho_A)^n \sim (C_n)(a)^{2d_n} \left\langle \mathcal{T}_n(u_1, 0)\tilde{\mathcal{T}}_n(v_1, 0)\right\rangle_{\mathcal{L}^n,\mathbb{C}} \quad (2.11)$$

Where C_n is a non-universal constant having $C_1 = 1$. The two-point function from above has a similar relation as in (A.38), but in the case of the twist fields they have scaling dimension $(d_n, \bar{d}_n)$. Knowing this, the two-point function of the twist fields is:

$$\left\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\right\rangle = |v_1 - u_1|^{-2d_n} \tag{2.12}$$

This relation will be important to compute entanglement entropy. In [11] they compute entanglement entropy for the infinite size system by taking the mapping $w \to (w-u_1)^{1/n}/(w-v_1)^{1/n}$. This maps all the n-sheeted Riemann surface $\mathcal{R}_n$ to the complex plane $\mathbb{C}$. Afterwards, they calculate the expectation value of the transformed stress tensor under conformal transformations on the n-sheeted surface $\mathcal{R}_n$. This expectation value has the following relation:

$$\langle T_i(w)\rangle_{\mathcal{R}_n} = \frac{\left\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)T_i(w)\right\rangle_{\mathcal{L}^n,\mathbb{C}}}{\left\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\right\rangle_{\mathcal{L}^n,\mathbb{C}}} \tag{2.13}$$

Using the above relation (2.13)and the Ward identity (A.35), Calabrese and Cardy the conformal dimension d_n was determined. In the end the entanglement entropy they got was:

$$S_A = \frac{c}{3}\log\left(\frac{l}{a}\right) + C_1' \tag{2.14}$$

Where $C_1' = -\partial_n C_n$ at $n = 1$. I will discuss the details of this calculation in next chapter.

Entanglement entropy in time-dependent states in CFT

In this part I will summarize from [1] the method to compute entanglement entropy for time-dependent states in the QFT, which in our case corresponds to CFT. First we consider a QFT with a time-dependent background, and as we know from QFT and quantum mechanics (QM), the states evolve with time by the time evolution operators. At a time t, the state is denoted by $|\Psi(t)\rangle$. For the explicit time-dependent Hamiltonian(t), the quantum state at an instant of time t is:

$$|\Psi(t)\rangle = T\exp\left(-i\int_{t_0}^{t} dt_1 H(t_1)\right)|\Psi(t_0)\rangle \tag{2.15}$$

Where T is the time ordered operator [2] and t_0 is the initial time. The ket and bra state are constructed by the path integrals:

$$\Psi(t,\phi_0(x)) = \int_{-\infty}^{t_1=t} [D\phi]e^{iS(\phi)}\delta(\phi(t,x)-\phi_0(x)) \tag{2.16}$$

$$\bar{\Psi}(t,\phi_0(x)) = \int_{t_1=t}^{\infty} [D\phi]e^{iS(\phi)}\delta(\phi(t,x)-\phi_0(x)) \tag{2.17}$$

Where we represent all the fields by ϕ. The two relations from above satisfy Schrödinger equation for the ket and bra case. The time-dependent density matrix is obtained by evolving in time the pure states of the quantum system. The

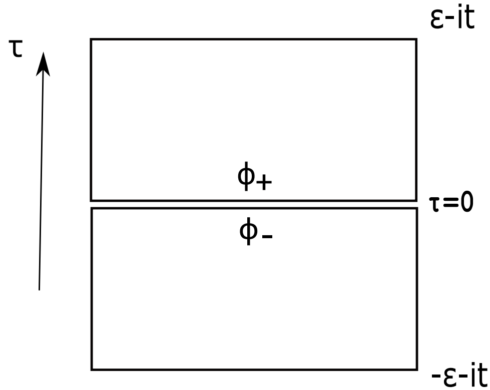


Figure 2.3: Illustration of the quantum path integral formulation of the reduced density matrix $[\rho_A]_{\{\phi_+\},\{\phi_-\}}$.

time dependent entanglement entropy requires the system to be divided into two subsystems at a certain instant of time t. Thus, the time-dependent Von Neumann entropy is:

$$S_A(t) = -tr(\rho_A(t)\log(\rho_A(t))) \tag{2.18}$$

Where $\rho_A(t)$ is the time dependent reduced density matrix, this is obtained by tracing the total density matrix along subsystem B at an instant of time t. Its quantum path integral formulation of the reduced density matrix is:

$$[\rho_A(t)]_{\{\phi_+\},\{\phi_-\}} = \frac{1}{Z_1}\int_{t=-\infty}^{t=\infty}[D\phi]e^{iS(\phi)}\prod_{x\in A}\delta(\phi(x,t+\epsilon)-\phi_+(x))\delta(\phi(x,t-\epsilon)-\phi_-(x)) \tag{2.19}$$

The infinitesimal factors ϵ are the damping factors used in such a way that the quantum path integral is absolutely convergent [16]. I inserted Figure (2.3) to illustrate the integral form of the reduced density matrix. Using the replica trick implies taking the product of n density matrices: $[\rho_A(t)]_{\phi_{1+}\phi_{1-}}...[\rho_A(t)]_{\phi_{n+}\phi_{n-}}$. In the static case I mentioned that the replica trick computes Rényi entropy, which in our case will depend of $tr(\rho_A(t))^n$. In order to get this trace it is necessary to assume that: $\phi_{1-} = \phi_{2+}, \phi_{2-} = \phi_{3+}, ...$ and $\phi_{n-} = \phi_{1+}$. This yields the partition function $Z_n(t)$ in the singular space-time manifold called $\mathcal{M}_n$. Moving from the manifold $\mathcal{M}$ of the replicated system to the complex plane $\mathbb{C}$, it is obtained:

$$tr(\rho_A(t))^n \sim C_n(a)^{2d_n}\left\langle \mathcal{T}_n(z_1(t),\bar{z}_1(t))\tilde{\mathcal{T}}_n(z_2(t),\bar{z}_2(t))\right\rangle \tag{2.20}$$

Where d_n is the scaling dimension. In this case it is expected that the twist fields depends of the complex conjugate because, as we will see in our computation of entanglement entropy in the CFT_2, the complex variables $z(t)$ are time dependent. These are functions that also depends of the branch points of subsystem A. This coordinate behaves as a analytic function $f(w)$ where $w = x + i\tau$, where ($\tau = -it$) is the imaginary time. In order to determine the relation between the conformal dimension d_n in terms of the number of copies of the system n and conformal charge c. The Ward identity will be applied to the correlation function: $\left\langle T(z)\mathcal{T}_n(z_1,\bar{z}_1)\tilde{\mathcal{T}}_n(z_2,\bar{z}_2)\right\rangle_{\mathcal{L}^n,\mathbb{C}}$.

2.2.2 Holographic method

In here I will discuss how to compute the entanglement entropy in (d+1)-dimensional conformal field theory (CFT_{d+1}) via the AdS/CFT correspondence. I will summarize from [17] the theoretical background of the holographic method having the time t fixed.

The AdS/CFT correspondence is useful to calculate entanglement entropy as a geometrical quantity in the AdS_{d+2} space, which must be equivalent to a specific system in the CFT. Using the Poincaré's patch of AdS_{d+2}, the metric is:

$$ds^2 = R^2 \frac{dz^2 - dx_0^2 + \sum_{i=1}^{d-1} x_i^2}{z^2} \tag{2.21}$$

According to this metric, the CFT_{d+1} is supposed to live in the boundary of AdS_{d+2} which is $R^{1,d}$ at $z \to 0$ spanned by the coordinates (x^0, x^i). Where the coordinate x^0 is the time t variable. The Poincaré metric has a point where it diverges, which is for $z \to 0$. For this reason we put a cut-off value by imposing the condition $z \geq a$. Using this cut-off value a implies that the boundary of AdS_{d+2} is situated on $z = a$ and it can be identified as the UV cut-off in the dual CFT.

Holographic Entanglement Entropy (static case)

Entanglement entropy can be calculated by using AdS/CFT. We know that in the CFT, entanglement entropy is well defined by dividing the quantum system into two subsystems A and B. This division is obtained by dividing a time slice $\mathcal{N}$ into two parts A and B in the CFT_{d+1}. Using the Poincaré patch, it is possible to take the time slice $\mathcal{N}$ equal to $\mathbb{R}^d$. On the other hand, the CFT_{d+1} lives on the boundary $z = a$ of the AdS_{d+2}. According to [17], the gravity dual is obtained by extending the division of the time slice $\mathcal{N}$ to the time slice $\mathcal{M}$ of the bulk.

Fixing the time in the Poincaré patch (2.21), the time slice of the bulk $\mathcal{M}$ is a (d+1)-dimensional hyperbolic plane. Also, it is possible to extend the boundary ∂A of subsystem A to a surface γ_A in the entire Euclidean manifold $\mathcal{M}$. In [12] they propose that the Von Neumann entropy satisfies the following area:

$$\boxed{S_A = \frac{A(\gamma_A)}{4G_N^{(d+2)}}} \tag{2.22}$$

Where $G_N^{(d+2)}$ is the (d+2)-dimensional gravitational constant and $A(\gamma_A)$ is the static d-dimensional minimal area of γ_A.

In this thesis I will compute entanglement entropy for the (1+1)-dimensional CFT, therefore I will use the area law for the restriction $d = 1$. It is known that the conformal charge c is related with the constant G_N^3 by the following relation:

$$\boxed{c = \frac{3R}{2G_N^{(3)}}} \tag{2.23}$$

Where R is the AdS_3 radius. Also we should mention that for $d = 1$, the minimal area of γ_A is the geodesic path length between two arbitrary points on the space-time. Therefore, the holographic method will consists in determining geodesics on a specific metric of space and time.

Covariant Prescription for Holographic Entanglement Entropy

In here I will review the covariant proposal of holographic entanglement entropy explained in [1]. Hubeny, Rangamani, and Takayanagi explains that in principle there shouldn't be any problem in generalizing the area law (2.22) for the time-dependent case. It is considered a time-dependent AdS/CFT, where the CFT is still in the boundary of the AdS space-time. The quantum states in the CFT will vary in time on a fixed background called $\partial\mathcal{M}$. In this case, it is considered a time-varying bulk geometry $\mathcal{M}$. In its boundary $\mathcal{M}$, it is possible to glue equal time slices. For this foliation it is considered that the factor of time t is involved in the time evolution of the field theory. This means that the fixed background is $\partial\mathcal{M} = \partial\mathcal{N}_t \times \mathbb{R}_t$.

As in the static case, we need to subdivide the total system into two parts A and B in order to compute the entanglement entropy. It is possible to define the region corresponding to subsystem A by time slices, where in [1] is denoted as $\mathcal{A}_t \in \partial\mathcal{N}_t$. We will take this region to compute entanglement entropy.

According to [1], for the covariant prescription of entanglement entropy must satisfy an area law similar to the static case. They propose four different covariant constructions of the surface to be used in the area law. In this thesis I will take the extremal surface $\mathcal{W}$, which has a saddle point of the area action. For the AdS_3, this extremized surface corresponds to a space-like geodesic through the bulk connecting all the points of $\partial\mathcal{A}_t$. The area law for this covariant prescription of entanglement entropy is:

$$\boxed{S_A(t) = \frac{A(\mathcal{W})}{4G_N^{(d+2)}}} \tag{2.24}$$

Where $A(\mathcal{W})$ is the extremized surface area on the AdS space.

In other words, for the 3-dimensional bulk the computation reduces in determining the geodesics on the time-dependent metric. The extremized path length can be determined by using Euler Lagrange equations, where the path length between two arbitrary points on the metric $g_{\mu\nu}$ is:

$$L_{\mathcal{W}}(t) = \int_1^2 \sqrt{g_{\mu\nu} dx^{\mu} dx^{\nu}} \tag{2.25}$$

Also, it is assumed that in the AdS_3/CFT_2 duality the conformal charge c is the same as in the static case (2.23).

Chapter 3

Entanglement Entropy Results in the CFT

In this chapter I will introduce the results of the time-dependent entanglement entropies by using the replica trick in the CFT. I will check that it is possible to fix the time t in the entanglement entropy and that it yields the static solution. Also, I will explain why the infinite or finite size of the quantum system at criticality, the entanglement entropy of subsystem A yields the same result as its complement subsystem B. I will also verify if this equality holds in the case of a infinite large system at finite temperature.

I will first review the computations of entanglement entropy previously done in [11] by using the replica trick in the static case. Therefore, the twist fields are time independent because the pure states of the physical system are not evolving with time.

After obtaining the static solution of entanglement entropy, I will consider that the quantum states evolve with time. This implies that also the entanglement entropy is dependent of time. I will compute entanglement entropy in 2D CFT for the real line (infinite size), on a circle (finite size) at zero temperature and the real line at finite temperature.

3.1 Entanglement entropy in CFT_2 for an infinite long system at zero temperature (static case)

Having the time fixed, the calculation of entanglement entropy is simplified by assuming $t = 0$. To compute entanglement entropy it is necessary to calculate the scaling dimension d_n in terms of the conformal dimensions h. Using the Replica trick to compute the trace $tr(\rho_A)^n$, it is necessary to determine the two point function $\left\langle \mathcal{T}_n(u_1, 0)\tilde{\mathcal{T}}_n(v_1, 0)\right\rangle_{\mathcal{L}^n,\mathbb{C}}$ from (2.11) by using a quantum path integral

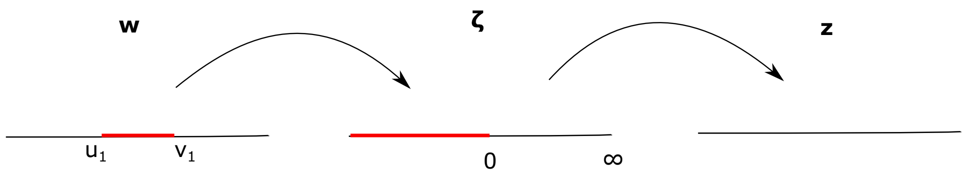


Figure 3.1: This figures shows the uniformizing transformation for the infinite size system. We have $w \to \zeta = (w-u_1)/(w-v_1)$ that maps the branch points to $(0,\infty)$. The uniformazing transformation corresponds to $\zeta \to z = \zeta^{1/n}$.

formulation. Luckily, as I explained in (A.37), in the CFT there is an advantage against the normal quantum field theory because the conformal symmetry can be used to simplify the calculations. For the infinite long system, I mentioned in the previous chapter that the two-point function of the twist fields is given by (2.12).

This section corresponds to the infinite size system at criticality. Thus, the conformal mapping to use for this system is: $w \to \zeta = \frac{w-u}{w-v}$, which maps the branch points to $(0,\infty)$. Also, it is required to map from the n-sheeted Riemann surface $R_{n,1}$ to the complex plane $\mathbb{C}$, this is done by taking the mapping $\zeta \to z = \zeta^{1/n}$, (see figure (3.1) for an illustration).

The next step is to use holomorphic component of the stress tensor $T(w)$, which is related to the stress tensor in the complex coordinate z. Using relation (A.33), this is:

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12} S(z,w) \tag{3.1}$$

The next step is to connect this relation with the 2-point function (2.12), so we must get first the expectation value over the n-sheeted surface $\mathcal{R}_{n,N}$ of the stress tensor $\langle T(w)\rangle_{\mathcal{R}_n}$. In that case we have that the expectation value $\langle T(z)\rangle_{\mathbb{C}} = 0$. Thus, we have:

$$\langle T(w)\rangle_{\mathcal{R}_{n,1}} = \frac{c(n^2-1)}{24n^2} \frac{(v_1-u_1)^2}{(w-u_1)^2(w-v_1)^2} \tag{3.2}$$

This was obtained by using the Schwarzian derivative (A.34), and this result is only for a individual sheet surface. To compute entanglement entropy we need the expectation value for the the n-sheet replicated surface. This is done by multiplying a factor of n to the result obtained in (3.2). Knowing this, we must remember that this is related to the correlation function (2.13). Therefore the correlation function of the stress tensor is:

$$\left\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)T_i(w)\right\rangle_{\mathcal{L}^{(n)},\mathbb{C}} = \frac{c(1-n^{-2})(v_1-u_1)}{24(w-u_1)^2(w-v_1)^2} \left\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\right\rangle_{\mathcal{L}^n,\mathbb{C}} \tag{3.3}$$

Where sub-index i for the stress tensor means that the result from (3.2) was for a single sheet surface. This is why we multiply by n this result when we want the solution for the n replicated system since the total stress tensor in that system will be the sum of the stress tensor of each individual system. From this correlation of function (3.3) for the energy tensor T we will use it as a reference in order to calculate the scaling dimensions. On the other hand, using the Ward identity (A.35) the correlation function of the stress tensor yields:

$$\left\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)T^n(w)\right\rangle_{\mathcal{L}^n,\mathbb{C}} = h_n\frac{(v_1-u_1)^2}{(w-u_1)^2(w-v_1)^2}\left\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\right\rangle_{\mathcal{L}^n,\mathbb{C}} \tag{3.4}$$

Where $h_n = \frac{c(n-n^{-1})}{24}$ is the conformal dimension, but this is only possible when the scaling dimension d_n is $d_n = 2h_n$. Therefore the scaling dimension is:

$$d_n = \frac{c(n-n^{-1})}{12} \tag{3.5}$$

In relation (3.4) we denoted $T^{(n)}(w)$ as the total energy-momentum tensor of the n replicated system. Having the scaling dimension in terms of the conformal charge and the number of sheet surfaces and inserting the two point function (2.12) in (2.11), the trace of ρ_A^n is:

$$tr(\rho_A)^n \sim (C_n)\left(\frac{v-u}{a}\right)^{-c(n-1/n)/6} \tag{3.6}$$

Using (2.8) and $l = v_1 - u_1$, which is the size of subsystem A, the Von Neumann entropy is:

$$S_A = \frac{c}{3}\log\left(\frac{l}{a}\right) + C_1' \tag{3.7}$$

where C_1' is the minus derivative respect to n of the constant C_n at $n = 1$, where $C_1 = 1$.

3.2 Covariant prescription of entanglement entropy in CFT_2 for an infinite long system at zero temperature

Now I proceed with the calculation of the time-dependent entanglement entropy. I discussed before that the quantum states evolve with time t implies a time dependence in the reduced density matrix $\rho_A(t)$ (2.19). The calculation is quite similar as in the static case, but in this case the twist fields depend of the conjugate of the complex variable z. Also, the complex conjugate can be used as an independent variable in this case. The scaling dimension $\bar{h}$ which corresponds to the complex conjugate of $\bar{z}$ is the same as the regular scaling dimension h. The correlation function of the stress tensor for the time-dependent case is:

$$\left\langle \mathcal{T}_n(s_1(\tau), \bar{s}_1(\tau))\tilde{\mathcal{T}}_n(s_2(\tau), \bar{s}_2(\tau))\right\rangle = (s_1(\tau) - s_2(\tau))^{-d_n}(\bar{s}_1(\tau) - \bar{s}_2(\tau))^{-d_n} \quad (3.8)$$

Where $s_1(\tau)$ and $s_2(\tau)$ are complex variables of the form $s_i(\tau) = x_i + i\tau_i$. The variable τ is the imaginary time $\tau = -it$, I will use this in the end of the computation to get the Lorentzian solution of entanglement entropy. For this time-dependent case, I am still working with the conformal mapping as in the static case, where $\zeta = \frac{w-s_1(\tau)}{w-s_2(\tau)}$ and $\zeta \to z = \zeta^{1/n}$. Even for the time dependent case, the Schwarzian derivative yields the same result as in the fixed time case. The correlation function of the stress tensor in our new notation is:

$$\begin{aligned}\left\langle \mathcal{T}_n(s_1(\tau), \bar{s}_1(\tau))\tilde{\mathcal{T}}_n(s_2(\tau), \bar{s}_2(\tau))\right\rangle_{\mathcal{L}^n,\mathbb{C}} =& \frac{c(1-n^{-2})(s_1(\tau)-s_2(\tau))}{24(w-s_1(\tau))^2(w-s_2(\tau))^2}\times \\ &\times \left\langle \mathcal{T}_n(s_1(\tau), \bar{s}_2(\tau))\tilde{\mathcal{T}}_n(s_2(\tau), \bar{s}_2(\tau))\right\rangle\end{aligned} \quad (3.9)$$

The scaling dimension for this time-dependent case is the same as in the static case (3.5). Substituting the $s_1(\tau)$ and $s_2(\tau)$ variables in (3.8) yields:

$$\left\langle \mathcal{T}_n(z_1(\tau), \bar{z}_1(\tau))\tilde{\mathcal{T}}_n(z_2(\tau), \bar{z}_2(\tau))\right\rangle = \left(\sqrt{(x_1-x_2)^2 + (\tau_1-\tau_2)^2}\right)^{-2d_n} \quad (3.10)$$

By substituting the conformal dimension d_n and (3.10) in the trace of the n replicated system density matrix from (2.20), the entanglement entropy is:

$$S_A(\tau) = \frac{c}{3}\log\left(\frac{\sqrt{(\Delta l)^2 + (\Delta\tau)^2}}{a}\right) + C_1' \quad (3.11)$$

Where $\Delta l = x_2 - x_1$ is the interval length of subsystem A and $\Delta\tau$ is the interval of imaginary time τ. This result is for the Euclidean version, so changing back to the Lorentzian time the entanglement entropy is:

$$S_A(t) = \frac{c}{3}\log\left(\frac{\sqrt{(\Delta l)^2 - (\Delta t)^2}}{a}\right) + C_1' \quad (3.12)$$

As we can see, there is no problem at all if we assume the fixed time in this solution, it yields the same result obtained in the static case, not to mention that C_1' would be the same as in the static case even for any instant of time. This is obvious since this calculation comes from the use of time evolution operators in the CFT.

3.3 Entanglement entropy in CFT_2 for a finite size of a system at zero temperature (static case)

In this section I will review the computation of entanglement entropy for a finite size from [11]. The finite length of the total system we denote it as L and we will still denote l as the size of the interval in subsystem A. In this case, we take the following conformal mapping: $w \to w = e^{i\frac{2\pi}{L}z}$, this corresponds to a cylinder where the branch cuts are oriented perpendicular to the axis. Remember that these branch cuts sew together to form the whole cylinder.

Now that we have the conformal mapping, I proceed to calculate the two-point function. In this case, the two-point function is different from the infinite size system because as it is explained in Appendix A, the primary fields satisfy a law under conformal transformations (see A.17). In this case the two point function of the twist fields transforms as follows:

$$\left\langle \mathcal{T}_n(z_1,\bar{z}_1)\tilde{\mathcal{T}}_n(z_2,\bar{z}_2)\right\rangle_{\mathcal{L}^n,\mathbb{C}} = \left|\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_2}\right|^{d_n} \left\langle \mathcal{T}_n(w_1,\bar{w}_1)\tilde{\mathcal{T}}_n(w_2,\bar{w}_2)\right\rangle_{\mathcal{L}^n,\mathbb{C}} \tag{3.13}$$

After using 2.12, the transformed two point function yields:

$$\left\langle \mathcal{T}_n(w_1,\bar{w}_1)\tilde{\mathcal{T}}_n(w_2,\bar{w}_2)\right\rangle_{\mathcal{L}^n,\mathbb{C}} = \left(\frac{L}{\pi}\sin\left(\frac{\pi}{L}(z_1-z_2)\right)\right)^{-2d_n} \tag{3.14}$$

Since the two point function transforms under a general conformal transformation, then the scaling dimension d_n is the same as in the infinite size system (3.5)

Having the two point function of the twist fields, we denote $l = z_1 - z_2$ as the interval length of subsystem A. This will be an equivalent version of computation for the finite temperature case. Having the scaling dimension value this leads us to use (2.11) in order to compute the trace of the reduced density matrix:

$$tr(\rho_A)^n \sim (C_n)a^{2d_n}\left\langle \mathcal{T}_n(z_1,\bar{z}_n)\tilde{\mathcal{T}}_n(z_2,\bar{z}_2)\right\rangle_{\mathcal{L}^n,\mathbb{C}} = (C_n)\left(\frac{L}{\pi a}\sin\left(\frac{\pi}{L}(z_1-z_2)\right)\right)^{-2d_n} \tag{3.15}$$

This lead to the static entanglement entropy in the CFT for the finite size.

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right) + C_1' \tag{3.16}$$

This solution tends to the real line entanglement entropy when the size of subsystem A is so small compared to the size of the system.

3.4 Covariant prescription of entanglement entropy in CFT_2 for a finite size of a system at zero temperature

For the time dependent case, using again the conformal mapping $w \to w = e^{i\frac{2\pi}{L}z}$. The branch cuts are oriented perpendicular to the axis of the cylinder, even these branch cuts are time dependent, these will sew together to form the whole cylinder. Even when the twist fields are time dependent, the scaling dimension d_n has the same value as in the infinite size system. The only thing that changes is the value of the two point function, so using the fact that the two point function transforms, we get in this case:

$$\left\langle \mathcal{T}_n(z_1(\tau), \bar{z}_1(\tau))\tilde{\mathcal{T}}_n(z_2(\tau), \bar{z}_2(\tau))\right\rangle_{\mathcal{L}^n,\mathbb{C}} = \left(\frac{L}{2\pi}\sqrt{2\cosh\left(\frac{2\pi\Delta\tau}{L}\right) - 2\cos\left(\frac{2\pi\Delta l}{L}\right)}\right)^{-2d_n} \tag{3.17}$$

I calculated the two point function (3.17) in a similar way to the static case only that this time I have $z_j = x_j + i\tau_j$ for $(j = 1, 2)$. This solution depends of the Euclidean time τ, so I used $\tau = -it$ to recover the Lorentzian time. Since the scaling dimension remains invariant, we can proceed to calculate the trace of the reduced density matrix $(\rho_A(t))^n$. For the Lorentzian time t I have:

$$tr(\rho_A(t))^n = C_n\left(\frac{L}{2\pi a}\sqrt{2\cos\left(\frac{2\pi\Delta t}{L}\right) - 2\cos\left(\frac{2\pi\Delta l}{L}\right)}\right)^{-c(n-n^{-1})/6} \tag{3.18}$$

Afterwards, I computed the Rényi entropy and took the limit $n \to 1$, so the Von Neumann entanglement entropy is:

$$S_A(t) = \frac{c}{3}\log\left(\frac{L}{2\pi a}\sqrt{2\cos\left(\frac{2\pi\Delta t}{L}\right) - 2\cos\left(\frac{2\pi\Delta l}{L}\right)}\right) + C_1' \tag{3.19}$$

We can observe that fixing the time, this result tends to the static solution (3.16). Something important to quote is that both results (3.16) and (3.19) are invariant by changing $l \to L - l$, this $L - l$ corresponds exactly to the length of subsystem B corresponding to the Hilbert space $\mathcal{H}_B$, this means that the entanglement entropy for both subsystems are equal $S_A(t) = S_B(t)$ even for the time dependent case (remember that B is the compliment of subsystem A). This raises the question, the fact that in the zero temperature case we have that both entanglement entropy are the same $S_A(t)$ and $S_B(t)$, will this equality hold for the finite temperature case? To answer this I will compute entanglement entropy in the CFT having the system at the temperature β^{-1}.

3.5 Entanglement entropy in CFT_2 of a system at finite temperature (static case)

This configuration where we have a finite temperature in the system will be the last configuration to study in the CFT. I will review the computation of the entanglement entropy by fixing the time and for the time-dependent case too. In this configuration I will consider that the system is infinitely large at temperature β^{-1}. The conformal mapping to this configuration is: $w \to w = e^{\frac{2\pi}{\beta}z}$. According to [11], this maps each sheet in the w plane to a infinitely long cylinder of circumference β. The cylinder will be sewn up by the branch cuts which are aligned to the parallel axis of the cylinder. The conformal dimension will yield again d_n as in (3.5). By using the fact that the two point function transforms by the conformal mapping, this yields:

$$\left\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\right\rangle_{\mathcal{L}^n,\mathbb{C}} = \left(\frac{\beta}{\pi}\sinh\left(\frac{\pi}{\beta}(v_1-u_1)\right)\right)^{-c(n-n^{-1})/6} \tag{3.20}$$

where u_1 and v_1 are the branch points, which defines the interval length ($l = v_1 - u_1$) of subsystem A. We can observe that the result from the finite size system at zero temperature is obtained by substituting $\beta = -iL$ in the two point function (3.14). The trace of the reduced density matrix of the n-replicated system is:

$$tr(\rho_A)^n = C_n\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi}{\beta}(v_1-u_1)\right)\right)^{-c(n-n^{-1})/6} \tag{3.21}$$

So by computing Rényi entropy and taking $n \to 1$ we get the entanglement entropy for subsystem A:

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right) + C_1' \tag{3.22}$$

This result of entanglement entropy was previously obtained in [11]. As we can see, this solution doesn't depend of a periodic function any more. This is important in the sense that if we compute entanglement entropy for the complement of subsystem A S_B then we don't get the same result as in S_A.

3.6 Covariant prescription entanglement entropy in CFT_2 of a system at finite temperature

In this section I will calculate the entanglement entropy for a system at finite temperature having the quantum states evolve with time. The conformal dimension d_n is the same as in the other systems. So in this sense, so far the computations of entanglement entropy in the CFT are simplified by this fact, and this is because of the symmetries regarding to the conformal transformations in the field theory. Computing the transformed version of the two point function we have:

$$\left\langle \mathcal{T}_n(z_1(\tau_1), \bar{z}_1(\tau_1)) \tilde{\mathcal{T}}_n(z_2(\tau_2), \bar{z}_2(\tau_2)) \right\rangle_{\mathcal{L}^n, \mathbb{C}} = \left(\frac{\beta}{2\pi} \sqrt{2\cosh\left(\frac{2\pi \Delta l}{\beta}\right) - 2\cos\left(\frac{2\pi(\Delta\tau)}{\beta}\right)} \right)^{-2d_n} \tag{3.23}$$

To calculate this two point function, I use $z_j = x_j + i\tau_j$ for $(j = 1, 2)$. Also we should mention that the interval length of subsystem A I denote it as $\Delta l = x_2 - x_1$ and the interval of Euclidean time as $\Delta\tau = \tau_2 - \tau_1$. Taking back to the Lorentzian time $\tau = -it$ and using the trace of the time-dependent reduced density matrix $\rho_A(t)$ from (2.20), the Von Neumann entanglement entropy is:

$$S_A(t) = \frac{c}{3} \log\left(\frac{\beta}{2\pi a} \sqrt{2\cosh\left(\frac{2\pi \Delta l}{\beta}\right) - 2\cosh\left(\frac{2\pi \Delta t}{\beta}\right)} \right) + C'_n \tag{3.24}$$

For the finite temperature case if we fix the time t we get the static result (3.22) which is expected. Also it is important to quote that since the entanglement entropy doesn't depend of any periodic functions as in the finite size case (3.19), the entanglement entropy for subsystem A differs from its compliment subsystem B. I finished with the computations done in the CFT side, now I shall proceed to the computation of entanglement entropy using holographic method.

Chapter 4

Entanglement Entropy using Holographic Method

Before calculating the time-dependent entanglement entropy, I will review first the computations for the static case. Those results were previously obtained in see [12]. These are obtained by minimizing the geodesic length on the metric of space-time. Also we will try to explain by illustrations of the geodesic lengths for three different metrics, of how entanglement entropy in subsystem A differs from the complement system B when the long system is at temperature β^{-1}. The metrics that I will use are: AdS_3 in Poincareé's and global coordinates, and the BTZ black hole for the time-dependent case. In the end we expect that these time-dependent results obtained by the covariant prescription yields the results obtained in chapter 3.

4.1 Entanglement entropy using Poincaré's metric for a fixed time

Before calculating the holographic entanglement entropy for the time-dependent case, we will have to compute the minimum geodesic path length using the Poincaré's metric:

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dr^2) \tag{4.1}$$

Where R is the radius of the AdS space, r is the radial variable.This coordinate patch is the one that by covering part of the space gives the half-space coordinatization of AdS space. The coordinate z divides the AdS space in two regions. Fixing the time in this coordinate patch corresponds to hyperbolic spaces in the Poincaré half plane metric. This is conformally equivalent to Minkowski space when $z \to 0$. This why it is said that the Poincaré space-time contains a conformal

Minkowski space at infinity, where under Poincaré's coordinates this corresponds precisely when $z \to 0$.

With this form of the metric, I proceed with the calculation of geodesics. I write down the expression for the interval length on this space-time in terms of the total derivative respect to "z" of the "r" (radial) variable, so we get:

$$ds^2 = \frac{R^2}{z^2}(1 + (r')^2)dz^2 \tag{4.2}$$

Where r' is the derivative of the variable $r(z)$ respect to "z". In order to calculate the geodesic path length, the following equation must be minimized:

$$L_{\gamma_A} = \int_{z_i}^{z_f} dz \frac{R}{z}\sqrt{1 + (r')^2} \tag{4.3}$$

In this case we have the UV cut-off denoted as "a" which is a really small value. To find the geodesic path length between the points z_f and z_i, we need to solve the Euler-Lagrange equation which in this case is:

$$\partial_z(\partial_{r'}\mathcal{L}) = 0 \tag{4.4}$$

Where the integrand of the geodesic path integral form (4.3) is denoted as $\mathcal{L}$. Solving this differential equation (4.4) we get the following conserved quantity respect to the variable z.

$$\partial_{r'}\mathcal{L} = C_1 \tag{4.5}$$

Where C_1 is a constant and solving the partial derivative to the path length we get the following result for r':

$$r' = \pm \frac{C_1 z}{\sqrt{R^2 - C_1^2 z^2}} \tag{4.6}$$

In this case the sign in (4.6) can be absorbed in the constant C_1. There should be a point in the path trajectory where the variable z gets a maximum value and that happens when $r' \to \infty$. That occurs when the denominator from relation (14) is equal to zero. This is the reason why it is assumed $C_1 > 0$. Therefore, the constant is:

$$C_1 = \frac{R}{z_*} \tag{4.7}$$

Where z_* is the maximum value of z of the geodesic path. The length of interval of subsystem A depends of the interval points r_f and r_i by the following relation: $l = r_f - r_i$. We need to find the expression of z_* in terms of the known constants a, r_f and r_i. The initial and final points in the z coordinates are precisely the cut-off value "a". We will need to integrate (4.6) in order to know the geodesics in the Poincaré's space-time. Solving the integral we get:

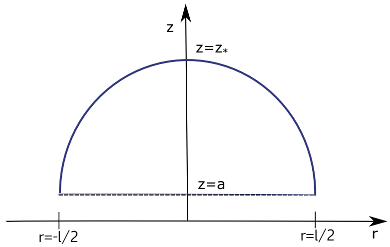


Figure 4.1: Illustration of the geodesic path γ_A for the Poincaré's having the time fixed.

$$r(z) = r_\epsilon + \left(\sqrt{z_*^2 - a} - \sqrt{z_*^2 - z^2}\right) \tag{4.8}$$

Where r_ϵ is the point in r where $z = a$. The equation above (4.8) corresponds to a half of a circle of radius $\frac{l}{2}$ but is translated by $z = a$(see figure 4.1). This means that the geodesic path in the Poincaré's metric is symmetric, which implies $z_f = a$ so we can do the following trick in the integral form of the radial coordinate $r(z)$ in order to obtain z_* in terms of the known interval length l of subsystem A. The trick consists that if we integrate from $z = a$ to $z = z_*$ and since the geodesic path is symmetric we have that the maximum value of z is reached for $r = 0$, which leads to the following equation:

$$l = 2\sqrt{z_*^2 - a^2} \tag{4.9}$$

In this relation, I can take the limit $a \to 0$ so this yields: $z_* = \frac{l}{2}$. This point is necessary in order to compute the minimum geodesic length, and basically this is the procedure for computing entanglement entropy by the holographic method at least for the other two metrics that I will use later. Now we can proceed with the integral from (4.3) where it is considered that the path is half of a circle. Also, using the fact that the geodesic path is symmetric for $-\frac{l}{2} \leqslant r \leqslant \frac{l}{2}$, the geodesic path length is:

$$L_{\gamma_A} = 2R \int_a^{z_*} dz \frac{1}{z} \sqrt{\frac{z_*^2}{z_*^2 - z^2}} \tag{4.10}$$

The extra 2 factor comes from the fact that the path is symmetric so we only need to integrate from a to the maximum value z_*. In order to solve this integral I made a change of variable $z = \frac{l}{2} \sin(s)$. If we observe the boundary condition, when $z = a$ this implies $\sin(s_\epsilon) = \frac{2a}{l}$. Also, we know that this value is small, actually it can be considered as a infinitesimal value, so approximately this is: $s_\epsilon = \frac{2a}{l}$. Therefore the integral we need to solve transforms into:

$$L_{\gamma_A} = 2R \int_{\frac{2a}{l}}^{\frac{\pi}{2}} ds \frac{1}{\sin(s)} \tag{4.11}$$

Using the fact that the cut-off value is an infinitesimal value, the geodesic path length is:

$$L_{\gamma_A} = 2R \log\left(\frac{l}{a}\right) \tag{4.12}$$

Since this computation is done by using the AdS_3/CFT_2 duality, then the $Area(\gamma_A)$ is precisely the minimum geodesic length in the space-time described by its metric. We can see so far that the behaviour of the geodesic length is the same as in the case of entanglement entropy we got for the infinite long quantum system case having the time fixed (3.7) except for the proportion factor $\frac{R}{2G_N^{(3)}}$. For the AdS_3 CFT_2 correspondence there is a known relation between the central conformal charge c with the radius R of AdS and gravitational constant G_N^3 from (2.23). Knowing this, the entanglement entropy is:

$$S_A = \frac{c}{3} \log\left(\frac{l}{a}\right) \tag{4.13}$$

Comparing this with the entanglement entropy from the CFT result, we can observe that it does not have the constant C_n'. This is quite natural because even in the CFT results this constant can be ignored because the cut-off value "$a \to 0$" is in the denominator. Therefore the logarithm part has a large value compared to the constant C_n'.

4.2 Covariant prescription of entanglement entropy using Poincaré's metric

Now that we saw how the holographic method of entanglement entropy works in the static case, now I proceed with the covariant prescription of entantlement. This requires working with the complete version of Poincaré's metric so I can't fix the time in this calculations for the geodesic length. The complete version of Poincaré's metric having the correct Lorentzian signature is: $ds^2 = \frac{R^2}{z^2}(dz^2 - dt^2 + dr^2)$. Using the same trick as in the case for the fixed time, I get the following relation for the geodesic path length:

$$L_{\mathcal{W}} = R \int_a^{z_f} dz \sqrt{1 - t'^2 + r'^2} \tag{4.14}$$

Where $r' = \partial_z r$ and $t' = \partial_z t$. We need to extremize the geodesic path length between two points of the z coordinate, the way to find those conditions consists in using the Euler-Lagrange equations, which in this case are:

$$\partial_z(\partial_{t'}L) = 0 \tag{4.15}$$
$$\partial_z(\partial_{r'}L) = 0 \tag{4.16}$$

These equations mean that there are two conserved quantities. Integrating both equations, this yields: $\partial_{t'}\mathcal{L} = C_1$ and $\partial_{r'}\mathcal{L} = C_2$ where $\mathcal{L} = R\sqrt{1 - t'^2 + r'^2}$. Solving the partial derivatives of the Lagrangian $\mathcal{L}$, one gets the following relations:

$$C_1 = \frac{-Rt'}{z\sqrt{1 - t'^2 + r'^2}} \tag{4.17}$$
$$C_2 = \frac{Rr'}{z\sqrt{1 - t'^2 + r'^2}} \tag{4.18}$$

From these two relations is not difficult to see that $\frac{t'}{r'} = -\frac{C_1}{C_2}$. Using this relation one gets the equations for the derivatives r' and t':

$$r' = \pm\frac{C_2 z}{\sqrt{R^2 - (C_2^2 - C_1^2)z^2}} \tag{4.19}$$
$$t' = \mp\frac{C_1 z}{\sqrt{R^2 - (C_2^2 - C_1^2)z^2}} \tag{4.20}$$

By looking these two equations, we know that if $C_1 = 0$ that would mean that the time is fixed into some constant t_0. If that is the case we can observe that we get the same relation for r' in the static case. It is natural to think that including the factor of time in this problem there should be a maximum value of z that implies $r' \to \infty$ and also $t' \to \infty$, using these conditions we get the following relation for the maximum value we denote again as z_*:

$$z_* = \frac{R}{\sqrt{C_2^2 - C_1^2}} \tag{4.21}$$

Which it only makes sense for $C_1 < C_2$, otherwise we get an imaginary number, the problem about this is that the results for entanglement entropy will yield into a complex expression. What we can do is to absorb the signs from (4.19) and (4.20) into the constants C_1 and C_2. Even by considering the factor of time in the Poincaré's metric we have the cut-off value "a". After manipulating the interval of integration in order to solve r' and t' in terms of the interval length of subsystem A denoted as Δl and the interval of time Δt, I get:

$$\Delta l = 2\int_a^{z_*} dz \frac{C_2 z}{\sqrt{R^2 - (C_2^2 - C_1^2)z^2}} \tag{4.22}$$

$$\Delta t = 2\int_a^{z_*} dz \frac{C_1 z}{\sqrt{R^2 - (C_2^2 - C_1^2)z^2}} \tag{4.23}$$

The factor of 2 in the integrals above comes from the fact that the geodesic path are symmetric too. Solving these integrals we get:

$$\Delta l = 2\frac{C_2}{\sqrt{C_2^2 - C_1^2}}\sqrt{z_*^2 - a^2} \tag{4.24}$$

$$\Delta t = 2\frac{C_1}{\sqrt{C_2^2 - C_1^2}}\sqrt{z_*^2 - a^2} \tag{4.25}$$

The cut-off value "a" is an infinitely small parameter where we can ignore it in (4.24) and (4.25). Using these we obtain the value: $z_* = \frac{\sqrt{\Delta l^2 - \Delta t^2}}{2}$ which is the main value to determine in order to compute the extremized geodesic length. After some calculations we get the integral form of the geodesic path length which we notice that it has the same integrand form as in the static case, the only difference is the maximum point z_* which in this case depends of the time interval Δt.

$$L_{\mathcal{W}} = 2R\int_a^{z_*} dz \frac{1}{z}\sqrt{\frac{z_*^2}{z_*^2 - z^2}} \tag{4.26}$$

The factor of 2 comes from the fact that the geodesic path is symmetric. As we did in the static case, we use the change of variables $z = z_* \sin(s)$. So we get:

$$L_{\mathcal{W}} = 2R\log\left(\frac{\csc(s_i) + \cot(s_i)}{\csc(s_*) + \cot(s_*)}\right) \tag{4.27}$$

Where s_i is the initial angle when $z = a$ and s_* is precisely the angle where the maximum value z_* is located which in this case we have that $s_* = \pi/2$. In the end when we substitute s_i, which depends of z_*, and s_f in (4.27) we finally obtain the geodesic path length:

$$L_{\mathcal{W}}(t) = 2R\log\left(\frac{\sqrt{\Delta l^2 - \Delta t^2}}{a}\right) \tag{4.28}$$

Therefore, by using relation (2.24) for holographic entanglement entropy in the time dependent case we get:

$$S_A(t) = \frac{c}{3}\log\left(\frac{\sqrt{\Delta l^2 - \Delta t^2}}{a}\right) \tag{4.29}$$

This is the same result obtained in the CFT for the infinitely long system, remember that we can ignore the constant C_n since the cut-off value makes the entanglement entropy to go high values by the logarithm term. So far, the covariant prescription of holography works pretty well and we know for sure that we don't have any restriction at all by fixing the time in order to get the entanglement entropy for subsystem A.

4.3 Entanglement entropy in the AdS_3 using global coordinates with fixed time

In this section we are going to solve the holographic entanglement entropy using the AdS_3 in global coordinates. These coordinates satisfies the following relation:

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -R^2 \tag{4.30}$$

Where R is the radius of the space. The coordinates that satisfies this relation are:

$$X_{-1} = R\cosh(\rho)\cos(T) \tag{4.31}$$
$$X_0 = R\cosh(\rho)\sin(T) \tag{4.32}$$
$$X_1 = R\sinh(\rho)\cos(\theta) \tag{4.33}$$
$$X_2 = R\sinh(\rho)\sin(\theta) \tag{4.34}$$

Using these coordinates we get that the metric for the AdS_3 space is just:

$$ds^2 = R^2(-\cosh^2(\rho)dT^2 + d\rho^2 + \sinh^2(\rho)d\theta^2) \tag{4.35}$$

From the metric we can see that for $\rho \to \infty$ the differential path length ds^2 diverges too. This is the reason that in the following computation we will define a cut-off value for the ρ coordinate which we will call it ρ_c, which is a large number. In this case, I denote T as the factor of time, but this coordinate doesn't have any units as we can see in (4.31) and (4.32). This time T works as an angle and in fact $T \in [0, 2\pi]$. We will avoid closed time-like curves, so we will take the universal cover $T \in \mathcal{R}$. Now we have to find the geodesic path length of the system, in order to compute the entropy. Computing the geodesic length it is required to integrate over all the space, but since we have a cut-off value and we are fixing the time "t" then the integral expression for the path length is:

$$L_{\gamma_A} = \int_{\rho_i}^{\rho_c} d\rho R\sqrt{1 + \sinh^2(\rho)\theta'^2} \tag{4.36}$$

where $\theta' = \partial_\rho\theta$. Using Euler Lagrange equation and noticing that the integrand doesn't depend of θ then we have a conserved quantity corresponding to $\partial_\theta L = C_1$ with some constant C_1. So we get the following relation:

$$C_1 = \frac{\sinh^2(\rho)\theta'}{\sqrt{1+\sinh^2(\rho)\theta'^2}} \tag{4.37}$$

Using this relation we get θ' as a function of ρ and C_1, so we get:

$$\theta'(\rho) = \frac{C_1}{\sinh(\rho)\sqrt{\sinh^2(\rho) - C_1^2}} \tag{4.38}$$

In this case this relation must be an absolute value, but the geodesic path can also be interpreted as the motion path of a massive particle and we could put the condition that the change in angular coordinate is changing positively. There should be a point for ρ coordinate in the geodesic path where this variable is minimum because for a fixed time in the AdS_3, we are working over a circumference of radius ρ_c defined on a hyperbolic plane. This means that the geodesics in this space are going to be closed curves which has a returning point at ρ_*. This minimum point of ρ is what the geodesic path length will depend in our computation. Since ρ_* is a minimum point, it must satisfy that $\theta' \to \infty$. With this condition we have $(C1 = \sinh(\rho_*))$. Inserting this relation into the expression for $\theta'(\rho)$ we get:

$$\theta'(\rho) = \frac{\sinh(\rho_*)}{\sinh(\rho)\sqrt{\sinh^2(\rho) - \sinh^2(\rho_*)}} \tag{4.39}$$

Inserting the equation above in the integral form of the path length, this yields:

$$L_{\gamma_A} = R\int_{\rho_i}^{\rho_c} d\rho \frac{\sinh(\rho)}{\sqrt{\sinh^2(\rho) - \sinh^2(\rho_*)}} \tag{4.40}$$

The next step is to get a relation between the initial and final points of the angular θ coordinate with the minimum radial point ρ_* and the cut-off value ρ_c. In order to do that we will manipulate the limits of integrations so we will have a certain angular difference that is related with the angle corresponding to the minimum radius ρ_*, these differences are:

$$\theta_f - \theta_* = \int_{\rho_*}^{\rho_c} d\rho \frac{\sinh(\rho_*)}{\sinh(\rho)\sqrt{\sinh^2(\rho) - \sinh^2(\rho_*)}} \tag{4.41}$$

$$\theta_* - \theta_i = \int_{\rho_*}^{\rho_c} d\rho \frac{\sinh(\rho_*)}{\sinh(\rho)\sqrt{\sinh^2(\rho) - \sinh^2(\rho_*)}} \tag{4.42}$$

Summing both relations (4.41) and (4.42) yields the angular difference $\Delta\theta = \theta_f - \theta_i$, so by solving the integral we get an equation relating the difference angle of the trajectory of the geodesic path and the ρ known parameters.

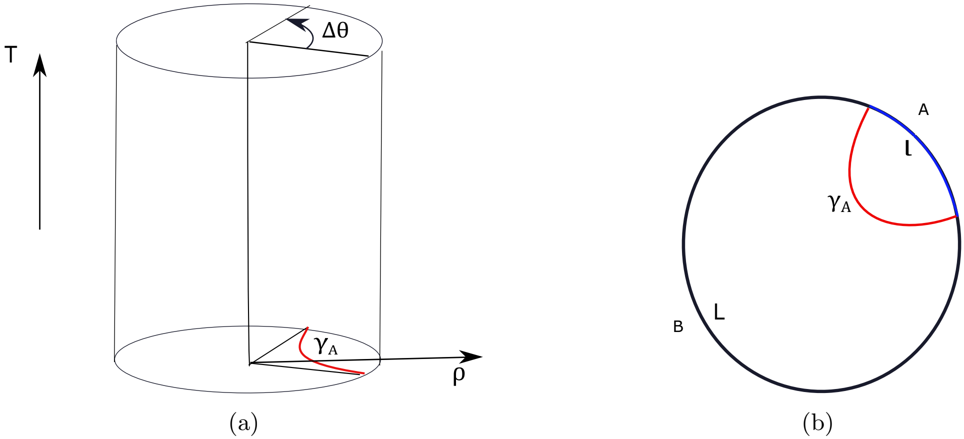


Figure 4.2: Figure(a): This is a plot of the AdS space where the red curve corresponds to the geodesic in the static case. Figure(b): Corresponds to a more detailed illustration of the geodesic path γ_A. The blue curve represents the interval corresponding to subsystem A and the black curve corresponds to the complement which is subsystem B.

$$\frac{\Delta\theta}{2} = -\tan^{-1}\left(\frac{\sqrt{2}\sinh(\rho_*)\cosh(\rho_c)}{\sqrt{-2sinh^2(\rho_*) + cosh(2\rho_c) - 1}}\right) + \frac{\pi}{2} \tag{4.43}$$

Also, we can write down the angular difference $\Delta\theta$ in terms of the total size length L of the total system and the size of subsystem A that I denoted as l. Since we have a circumference over a hyperbolic plane, then we can obtain another relation of the interval length l by integrating the metric having fixed the radial coordinate $\rho = \rho_c$. Also, a similar procedure can be done in order to have an expression for the length of the total system L. As we can see in figure (4.2), the geodesic path is just over the hyperbolic disk defined by the AdS_3 space when having the time T fixed, in this figure we can illustrate the fact that the observer is only accessible to the subsystem A and cannot receive any signals from B, in some sense it is similar case as what it happens in a black hole. After integrating the metric by fixing $\rho = \rho_c$ we obtain the boundary condition:

$$\Delta\theta = \frac{2\pi l}{L} \tag{4.44}$$

Using the expression for $\Delta\theta$ from (4.43) and using the fact that the cut-off value $\rho_c >> 1$ we get the relation for the minimum radial value ρ_*:

$$\sinh(\rho_*) = \cot\left(\frac{\Delta\theta}{2}\right) \tag{4.45}$$

Using (4.40), where the geodesic path γ_A begins the trajectory for $\rho_i = \rho_c$ and using the fact that this geodesic is symmetrical we get that the length of this geodesic is:

$$L_{\gamma_A} = 2R\log\left(\frac{\sqrt{\cosh(2\rho_c) - 2\sinh^2(\rho_c) - 1} + \sqrt{2}\cosh(\rho_c)}{\sqrt{2}\cosh(\rho_*)}\right) \tag{4.46}$$

Since we have $\rho_c >> 1$, we approximated $\cosh(\rho_c) \approx e^{\rho_c}$ and $\sinh(\rho_c) \approx e^{\rho_c}$. Inserting equation (4.45) this yields the following:

$$L_{\gamma_A} \approx 2R\log\left(2\ e^{\rho_c}\sin\left(\frac{\pi l}{L}\right)\right) \tag{4.47}$$

We can observe that this result of geodesic path length has a similar behaviour as the entanglement entropy result we did for the fixed time finite size system in the CFT. So again using the Area law (2.22) and the relation of conformal charge (2.23) with the gravitational constant G_N^3, the entanglement entropy is:

$$S_A = \frac{c}{3}\log\left(2e^{\rho_c}sin\left(\frac{\pi l}{L}\right)\right) \tag{4.48}$$

If we want to know what is exactly the cut-off value ρ_c then we should use the limit when l is really small compared to the total size length of the system "L". By doing this we can compare it with the one obtained in Poincaré case (4.13) This implies that $\rho_c = \frac{L}{2\pi a}$ where "a" is the cut-off value in the Poincaré's entanglement entropy solution. So in the end we get:

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}sin\left(\frac{\pi l}{L}\right)\right) \tag{4.49}$$

Which is in fact the entropy calculated for the finite size system we did in the CFT side (4.13). So far we can see how the duality works, which basically we have a certain quantum system configuration in the CFT and its entanglement entropy is equivalent into a particular geometry of space-time in the gravity side. This is important since as we have mentioned before, quantum gravity is what precisely consists about, determining the corresponding geometry in the gravity side for a certain field theory in the boundary. In this case it seems that the finite size system at zero temperature is dual to a geometry of AdS space. If we go back in our discussion in (chapter 4) about the fact that this solution of entanglement entropy for subsystem A satisfies $S_A = S_B$, this would mean that the geodesic length for subsystem B is basically the same as γ_A. Physically speaking this means that whenever the system is at zero temperature, the systems will always be in a pure state, but this changes when temperature is present in the system.

4.4 Covariant prescription of entanglement entropy in the AdS_3 using global coordinates

In this section I will compute the entanglement entropy including the factor of time T. In other words we have that the metric is of the form of relation (4.35). The geodesic path length can be written in the following form:

$$L_{\gamma_A} = R\int_{\rho_i}^{\rho_f} d\rho\sqrt{-\cosh^2(\rho)T'^2 + \sinh^2(\rho)d\theta^2 + 1} \tag{4.50}$$

Using the Euler-Lagrange equation, I will extremize this length. There are two quantities that are conserved since the geodesic path doesn't depend of the coordinates θ and T explicitly. So I have:

$$C_1 = \frac{\sinh^2(\rho)\theta'}{\sqrt{\sinh^2(\rho)\theta'^2 - cosh^2(\rho)T'^2 + 1}} \tag{4.51}$$

$$C_2 = \frac{-\cosh^2(\rho)T'}{\sinh^2(\rho)\theta'^2 - \cosh^2(\rho)T'^2 + 1} \tag{4.52}$$

From these two constants, I have the relation between θ' and T' which is: ($T' = -\frac{C_2}{C_1}\tanh^2(\rho)\theta'$). From this, I obtain the expressions the corresponding for θ' and T', so in the end I will have to integrate them after getting both relations:

$$\theta' = \frac{C_1}{\sinh(\rho)\sqrt{\sinh^2(\rho) - C_1^2 + C_2^2 tanh^2(\rho)}} \tag{4.53}$$

$$T' = -\frac{C_2\tanh(\rho)}{\cosh(\rho)\sqrt{\sinh^2(\rho) - C_1^2 + C_2^2 tanh^2(\rho))}} \tag{4.54}$$

Doing the same procedure as in the static case, I will find a specific value for ρ corresponding to the geodesic path, we will call it ρ_* as in the static solution. Also we should quote that we can absorb the sign in one of the constants C_1 or C_2. The fact that in this complete version of AdS_3 we have a point in the path where ρ is minimum, this means that the derivatives $\theta'(\rho_*)$ and $T'(\rho_*)$ tends to ∞. This leads to the following condition for the constant C_1:

$$C_1^2 = \sinh^2(\rho_*) + tanh^2(\rho_*)C_2^2 \tag{4.55}$$

Now after doing something similar as in the infinite long system time-dependent case, I work out again the limits of integration so we have the following relations for $\Delta\theta$ and time interval ΔT:

$$\frac{\Delta\theta}{2} = \int_{\rho_*}^{\rho_c} d\rho \frac{C_1}{\sinh(\rho)\sqrt{\sinh^2(\rho) - \sinh^2(\rho_*) + (\tanh^2(\rho) - \tanh^2(\rho_*))C_2^2}} \tag{4.56}$$

$$\frac{\Delta T}{2} = \int_{\rho_*}^{\rho_c} d\rho \frac{C_2 sinh(\rho)}{cosh^2(\rho)\sqrt{sinh^2(\rho) - sinh^2(\rho*) + (\tanh^2(\rho) - \tanh^2(\rho_*))C_2^2}} \tag{4.57}$$

In this section $\Delta\theta$ and ΔT are the angular and time difference between the corresponding initial point and last point of the geodesic path. Solving these integrals and using the fact that the cut-off is really large $\rho_c >> 1$, I get:

$$\frac{\Delta\theta}{2} = \tan^{-1}\left(\frac{\sqrt{cosh^2(\rho_*) + C_2^2}}{cosh(\rho_*)sinh(\rho_*)}\right) \tag{4.58}$$

$$\frac{\Delta T}{2} = \tan^{-1}\left(\frac{C_2}{cosh^2(\rho_*)}\right) \tag{4.59}$$

I have used already the fact that the cut-off value $\rho_c >> 1$ before computing the extremized geodesic path length because in the end, this will be a leading term compared to the other parameters. Now we can ask ourselves if the angular difference has the same relation as in the static case. In order to calculate it I will have to compute a path integral using the metric for global coordinates. Since l is the size of the interval between the initial and the last points of the geodesic path length, I will have to integrate the square root of the metric having $\rho = \rho_c$ fixed between the points θ_i and θ_f. Also, if I want the expression for the size of the whole system L, integrating from 0 to 2π, I have:

$$l = \int_{\theta_i}^{\theta_f} d\theta R\sqrt{\sinh^2(\rho_c) - \tanh^2(\rho_c)\sinh^2(\rho_c)\frac{C_2^2}{C_1^2}} \tag{4.60}$$

$$L = \int_{0}^{2\pi} d\theta R\sqrt{\sinh^2(\rho_c) - \tanh^2(\rho_c)\sinh^2(\rho_c)\frac{C_2^2}{C_1^2}} \tag{4.61}$$

So the size length l of subsystem A and the length L of the total system are:

$$l = \frac{R\Delta\theta\sqrt{C_1^2\sinh^2(\rho_c) - C_2^2\tanh^2(\rho_c)\sinh^2(\rho_c)}}{C_1} \tag{4.62}$$

$$L = \frac{2\pi R\sqrt{C_1^2\sinh^2(\rho_c) - C_2^2\tanh^2(\rho_c)\sinh^2(\rho_c)}}{C_1} \tag{4.63}$$

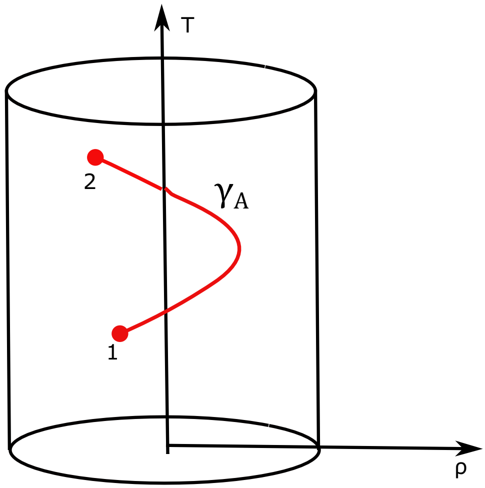


Figure 4.3: Geodesic path in the AdS_3 metric. The points 1 and 2 corresponds to the initial and final points respectively.

Dividing both terms we get that $\Delta\theta = \frac{2\pi l}{L}$, which is the same result as in the static solution.

Using (4.59) we get the constant C_2 in terms of the minimum radius ρ_* and the time interval T.

$$C_2 = cosh^2(\rho_*)\tan\left(\frac{\Delta T}{2}\right) \tag{4.64}$$

Now that we have this value, I need to find the relation for the minimum radius ρ_* in terms of the boundary conditions of this system. Using (4.58) we get:

$$\sinh(\rho_*) = \frac{1}{cos\left(\frac{\Delta T}{2}\right)\sqrt{\tan^2\left(\frac{\Delta\theta}{2}\right) - \tan^2\left(\frac{\Delta T}{2}\right)}} \tag{4.65}$$

The geodesic path is symmetric and is a closed curve which connects two points from the boundary of the infinite cylinder of radius ρ_c. As we can see in (figure 4.3) the projection of the geodesic path in the hyperbolic plane would correspond to the curve that we had in the static case.

$$L_{\mathcal{W}}(T) = 2R\int_{\rho_*}^{\rho_c} d\rho \frac{sinh(\rho)}{\sqrt{sinh^2(\rho) - \sinh^2(\rho_*) + (\tanh^2(\rho) - \tanh^2(\rho_*))C_2^2}} \tag{4.66}$$

This geodesic length after some difficult computation and using the equation for the minimum radial value ρ_* yields:

$$L_{\mathcal{W}}(T) = 2R\log\left(2e^{\rho_c} cos\left(\frac{\Delta T}{2}\right)\cos\left(\frac{\Delta\theta}{2}\right)\sqrt{\tan^2\left(\frac{\Delta\theta}{2}\right) - \tan^2\left(\frac{\Delta t}{2}\right)}\right) \quad (4.67)$$

After using some trigonometric identities and using (2.24), we get that the time-dependent holographic entanglement entropy for using AdS_3 in global coordinates is:

$$S_A(T) = \frac{c}{3}\log\left(e^{\rho_c}\sqrt{2\cos(\Delta T) - 2\cos(\Delta\theta)}\right) \quad (4.68)$$

We can observe that I got again this parameter e^{ρ_c} that depends of the cut-off value. This value was $e^{\frac{L}{2\pi a}}$, and this shouldn't change for this time-dependent case since is only a constant depending of the cut-off value "a". Also, we can see that the solution I got for entanglement entropy for subsystem A depends in a non-dimensional time, this is because of the form is constructed the metric that we used in the AdS_3 metric. If we want to express this in terms of the Lorentzian time t, we must use: $T = \frac{2\pi\Delta t}{L}$. We don't have to worry about taking a Wick rotation since we have already used the correct Lorentzian signature in the metric. By taking the change of variable in time and using the angular difference $\Delta\theta$ in terms of the size of interval of subsystem A, we have:

$$S_A(t) = \frac{c}{3}\log\left(\frac{L}{2\pi a}\sqrt{2\cos\left(\frac{2\pi\Delta t}{L}\right) - 2\cos\left(\frac{2\pi\Delta l}{L}\right)}\right) \quad (4.69)$$

This result is exactly the same as the one we computed in the CFT for finite size quantum system at zero temperature. The fact that this entropy is invariant under the transformation $l \to L - l$, in the holographic sense this means that the geodesic path corresponding to the complement subsystem B is equal to the geodesic path of subsystem A in the AdS_3 space-time. The question if this holds true for every well defined metric, where one can divide the Hilbert space of a system in two subsystems A and B and the entanglement entropy for both subsystems are equal is already answered when we calculated the entanglement entropy for a infinite long system at finite temperature in the CFT, where we basically obtained that $S_A(t) = S_B(t)$. What would the corresponding geometry for this case be in the gravity side? The main candidate is the BTZ black hole which is our next computation and we will check the interesting things that happens with the geodesic paths corresponding to subsystem A and subsystem B.

4.5 Entanglement entropy for the static BTZ black hole

A BTZ black hole is the black hole solution having a negative cosmological constant. This has similar properties to the 3+1 dimensional black hole.

In [12], they discuss this case of entanglement entropy in the static Euclidean BTZ black hole, where they concluded that this geometric prescription is dual to the quantum system at a certain temperature β^{-1}. We will compute again this static case to illustrate what happens to the geodesic path depending of certain values of the system we have. In order to proceed with the calculation of entropies we must use the metric corresponding to the Euclidean BTZ black hole which is:

$$ds^2 = \frac{(r^2 - r_+^2)}{R^2} d\tau^2 + \frac{R^2}{(r^2 - r_+^2)} dr^2 + r^2 d\varphi^2 \tag{4.70}$$

Where the euclidean time is compactified as $\tau \sim \tau + \frac{2\pi R^2}{r_+}$ in order to have a smooth geometry and also we have the periodicity $\phi \sim \phi + 2\pi$. Looking at its boundary, we obtain the relation $\frac{\beta}{L} = \frac{R}{r_+} << 1$ between the BTZ black hole and the CFT. Since we want to compute the geodesic path length for the static case, we fix the euclidean time to some constant $\tau = \tau_0$. The integral form of the geodesic path length L_{γ_A} is obtained by using the static version of the metric (4.70), so we have:

$$L_{\gamma_A} = \int_{r_i}^{r_f} dr \sqrt{\frac{R^2}{r^2 - r_+^2} + r^2 \varphi'^2} \tag{4.71}$$

Where φ' is just $\partial_r \varphi$. Before starting computing this, we will write down the difference angle $\Delta\varphi$ which is the difference between the final and initial angle. To do this we notice that for this static metric, we will have a torus, one of the circumference is of radius r_+ and the other will have the radius corresponding to the cut-off for this system. The geodesic path will be similar to the one we calculated for the AdS_3 global coordinate static case. Only that this time one has to be more careful because the geodesic path may enclose the horizon region. Fixing $r = r_c$ where r_c is the cut-off radius, we calculate the length l of subsystem A and the length of the whole system L by integrating ds.

$$l = r_c \Delta\varphi \tag{4.72}$$

$$L = 2\pi r_c \tag{4.73}$$

$$\Delta\varphi = \frac{2\pi l}{L} \tag{4.74}$$

Relation (4.74) is obtained by dividing (4.72) with (4.73). Using Euler-Lagrange equation in order to find geodesics for this metric, we get an expression for φ' which is:

$$\varphi' = \frac{C_1 R}{r\sqrt{(r^2 - r_+^2)(r^2 - C_1^2)}} \tag{4.75}$$

Where C_1 is the constant of motion related with the Euler-Lagrange equation for the φ coordinate. The geodesic path has a point where the radial coordinate is minimum. This is satisfied when $\varphi(r_*) \to \infty$, so this happens only when:

$$r_* = C_1 \tag{4.76}$$

Why not $r_* = r_+$? The answer is that every point in the radial coordinate r corresponding to the geodesic path there should be a value of r which is lower than the cut-off value but is higher than the r_+, since this is the point of the horizon of Euclidean BTZ black hole. The geodesic path never reaches the point of event horizon $r = r_+$, but it may surround it very close to it, that is why (4.76) is the correct expression for r_*. Using a similar procedure as we did in the static solution of AdS_3 global coordinates we get:

$$\frac{\Delta\varphi}{2} = \int_{r_*}^{r_c} dr \frac{r_* R}{r\sqrt{(r^2 - r_+^2)(r^2 - r_*^2)}} \tag{4.77}$$

Solving this integral, we get a dependence of Appell hypergeometric function of two variables, the integral yields $\frac{Rr_*}{2r^2} F_1(1; \frac{1}{2}, \frac{1}{2}; 2; \frac{r_+^2}{r^2}, \frac{r_*^2}{r^2})$, but when we evaluate for $r = r_c$ and $r = r_*$ and using the fact that $r_c >> 1$ we get:

$$\frac{\Delta\varphi}{2} = \frac{R}{r_+} tanh^{-1}\left(\frac{r_+}{r_*}\right) \tag{4.78}$$

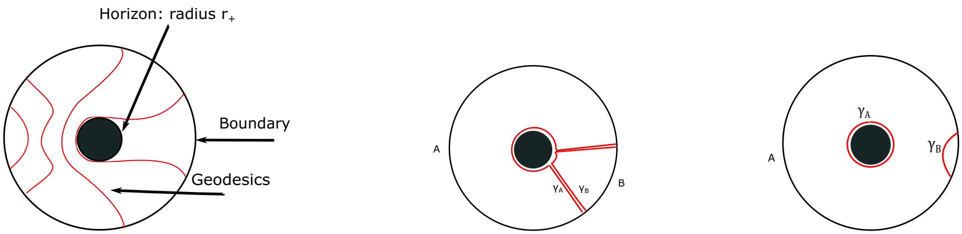
This yields since $F_1(1; \frac{1}{2}, \frac{1}{2}; 2; \frac{r_+^2}{r_*^2}, 1) = \frac{2r_*}{r_+} \tanh^{-1}\left(\frac{r_+}{r_*}\right)$ and for the case when $r = r_c$, the term of the result of the integral vanishes so basically (4.78) is enough for our computation of entanglement entropy. From equation above we can get the expression for the minimum radius r_* which is necessary in order to compute the geodesic length in terms of known parameters. We have:

$$r_* = r_+ \coth\left(\frac{\pi l}{\beta}\right) \tag{4.79}$$

Where the inverse temperature is $\beta = \frac{RL}{r_+}$. The integral corresponding to the geodesic path length yields:

$$L_{\gamma_A} = 2R\log\left(\frac{\sqrt{r_c^2 - r_+^2} + \sqrt{r_c^2 - r_*^2}}{r_*^2 - r_+^2}\right) \tag{4.80}$$

Using the fact $r_c >> 1$ and using relation (4.79) we get:



(a) (b) (c)

Figure 4.4: Figure(a): Geodesics or minimal surfaces γ_A in the BTZ black hole for various size of A. Figure(b): Illustration of how γ_A and γ_B surrounds differently the event horizon. Fiugre(c): This case is when when subsystem A is almost as large as the total system.

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$$L_{\gamma_A} = 2R\log\left(\frac{2r_c}{r_+}\sinh\left(\frac{\pi l}{\beta}\right)\right) \tag{4.81}$$

Using the transformation $r = r_+\cosh(\rho)$, which is useful to transform the metric from BTZ black hole to a similar Euclidean version of global AdS_3, and using the limit for a short length interval l of subsystem A, we have that $r_c \approx \frac{r_+\beta}{2\pi a}$, where a is the cut-off and is a really small value value $a \to 0$. In the end, we use the relation corresponding to the central charge of the CFT (2.23) so we get the solution:

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right) \tag{4.82}$$

Now we will discuss about the different geodesic path and its corresponding lengths in this metric of BTZ black hole (see figure 4.4 for an illustration). In (fig 4.4a) we can observe the form of the geodesic paths having different sizes of subsystem A, there is one of them that looks like the geodesic in the AdS_3 space-time, that corresponds only when the size l of subsystem A is a small value, if that happens compared to the inverse of temperature β we get $S_A = \frac{c}{3}\log\left(\frac{l}{a}\right)$. This actually has the same behaviour as the infinite long system at zero temperature, this makes sense since we consider $l << \beta$ is another way of saying that the temperature $\beta^{-1} << 1$. In the case of those geodesics that surrounds the BTZ black hole corresponds to those sizes of subsystem A that are bigger than those from subsystem B.

In (figure 4.4b) we illustrate that when the size l of subsystem A is comparable with the corresponding size of subsystem B we have the case that the path γ_A surrounds the btz black hole, but γ_B doesn't, this illustrates and explains the reason of why the entanglement entropies corresponding to those subsystems are different. We notice that in the gravity side, the one that is responsible that both geodesic paths are different is because we have a event horizon located at r_+ in this space-time. In the CFT the reason of why it differs would be that since the system is at a finite temperature this implies that the system is no longer in a pure state, in fact it is in a mixture state this is the reason of why $S_A \neq S_B$.

The last case we have is from (figure 4.4c), in here we have the case when the geodesic path encloses the BTZ black hole. This is the case if the size $l \sim L$, where the size corresponding to subsystem B is a really small number. So in this sense the corresponding geodesic path γ_B is the same as in the AdS_3. Now we will see if using the whole version of the metric of BTZ black holes yields the same result as we obtained in the CFT (3.24), in order to check that the covariant prescription works.

4.6 Covariant prescription of entanglement entropy for the BTZ black hole

Now I shall proceed with the last computation which is considering the factor of time in the BTZ black hole. First, I will begin with the computation using the Euclidean version of this metric just for simplifying the calculation. Afterwards, I will make a Wick rotation in order to recover the Lorentzian signature. In order to calculate the geodesic path length in this metric, I express this metric in terms of only derivatives of our coordinates $\varphi = \partial_r \varphi$ and $\tau' = \partial_r \tau$. So the geodesic path length is:

$$L_{\gamma_A} = \int_{r_i}^{r_f} dr \sqrt{f(r)\tau'^2 + \frac{1}{f(r)} + r^2\varphi'^2} \tag{4.83}$$

where I denoted $f(r) = \frac{r^2 - r_+^2}{R^2}$ The corresponding Lagrangian doesn't depend explicitly of φ and τ, only of their corresponding derivatives. This means that we have two conserved quantities, in other words:

$$C_1 = \frac{f(r)\tau'}{\sqrt{f(r)d\tau'^2 + \frac{1}{f(r)} + r^2\varphi'^2}} \tag{4.84}$$

$$C_2 = \frac{r^2\varphi'}{\sqrt{f(r)\tau'^2 + \frac{1}{f(r)} + r^2\varphi'^2}} \tag{4.85}$$

From these two relations we get the equations of the partial derivatives of our coordinates. We can observe that $\tau' = \frac{C_1 r^2}{C_2 f(r)}\varphi'$, so using this relation we can get the equalities for the derivatives:

$$\varphi'(r) = \frac{C_2}{r\sqrt{f(r)(r^2 - C_2^2) - r^2 C_1^2}} \tag{4.86}$$

$$\tau'(r) = \frac{C_1 r}{f(r)\sqrt{f(r)(r^2 - C_2^2) - r^2 C_1^2}} \tag{4.87}$$

There should be a minimum value of r where we denote again r_*, since the geodesics will be symmetric and will surround the BTZ black hole as in the static case. This means that both φ' and τ' diverges at this point. This condition yields an equation relating the two constants C_1 and C_2:

$$C_1^2 = \frac{f(r_*)(r_*^2 - C_2^2)}{r_*^2} \tag{4.88}$$

By integrating both (4.86) and (btzt5) we can get the angular and time intervals in function of the constant C_2 and r_*. So using the expression of f(r) we get:

$$\Delta\tau = \frac{2R^2}{r_*}\int_{r_*}^{r_c} dr \frac{r\sqrt{(r_*^2 - r_+^2)(r_*^2 - C_2^2)}}{(r^2 - r_+^2)\sqrt{(r^2 - r_*^2)(r^2 - \frac{r_+^2 C_2^2}{r_*^2})}} \tag{4.89}$$

$$\Delta\varphi = 2R\int_{r_*}^{r_c} dr \frac{C_2}{r\sqrt{(r^2 - r_*^2)(r^2 - \frac{r_+^2 C_2^2}{r_*^2})}} \tag{4.90}$$

These integrals yields the following by considering the limit of cut-off value $r_c >> 1$:

$$\Delta\tau = \frac{2R^2}{r_+}\tan^{-1}\left(\frac{r_+}{r_*}\sqrt{\frac{r_*^2 - C_2^2}{r_*^2 - r_+^2}}\right) \tag{4.91}$$

$$\Delta\varphi = \frac{2R}{r_+}\tanh^{-1}\left(\frac{r_+ C_2}{r_*^2}\right) \tag{4.92}$$

From (4.92) we get the value of constant C_2 in terms of r_*, the radius of BTZ black hole r_+, the space radius R and angular difference $\Delta\varphi$.

$$C_2 = \frac{r_*^2}{r_+} tanh\left(\frac{\Delta\varphi r_+}{2R}\right) \tag{4.93}$$

Using this value of C_2 and using equation (4.91) we finally get the relation between the minimum radial distance r_* in terms of the known parameters Δl and $\Delta\tau$:

$$r_* = \frac{r_+ \sec\left(\frac{\Delta\tau r_+}{2R^2}\right)}{\sqrt{\tan^2\left(\frac{\Delta\tau r_+}{2R^2}\right) + \tanh^2\left(\frac{\Delta\varphi r_+}{2R}\right)}} \tag{4.94}$$

Now that i have gotten the minimum point of radial coordinate r_*, I can insert this along with the conserved quantities C_1 and C_2 in the integral form of the geodesic path length, this yields:

$$L_{\gamma_A} = 2R\int_{r_*}^{r_c} dr \frac{r}{\sqrt{(r^2 - r_*^2)(r^2 - \frac{r_+^2 C_2^2}{r_*^2})}} \tag{4.95}$$

Again considering the limit $r_c >> 1$ we basically have:

$$L_{\gamma_A} = 2R\log\left(\frac{2r_c}{\sqrt{r_*^2 - \frac{r_+^2 C_2^2}{r_*^2}}}\right) \tag{4.96}$$

Now from here we can see that the extremized length has a similar behaviour as in the Euclidean result of time-dependent entanglement entropy we got in the CFT for the finite temperature case. Nowe we can recover The Lorentzian case for this problem by using Wick rotation $\tau = -it$ where t is the Lorentzian time. Using the area law, and the conformal charge we get:

$$S_A(t) = \frac{c}{3}\log\left(\frac{r_c}{r_+}\sqrt{2\cosh\left(\frac{\Delta\varphi r_+}{R}\right) - 2\cosh\left(\frac{\Delta t r_+}{R^2}\right)}\right) \tag{4.97}$$

It seems that this is not exactly the same result as in the CFT (3.24) since we have a dependence of the BTZ black hole radius r_+ and the AdS radius R. However we haven't considered yet the fact that one can get the following relation: $\beta = \frac{2\pi R^2}{r_+}$. This expression is found by calculating the deficit angles using the Lorentzian version of BTZ black hole, which is the responsible for the conical singularity of this space-time. Using the expression for $\Delta\varphi$, β and using the same value for the cut-off radius $r_c \approx \frac{r_+\beta}{2\pi a}$ that we had in the static solution, we finally get:

$$S_A(t) = \frac{c}{3}\log\left(\frac{\beta}{2\pi a}\sqrt{2\cosh\left(\frac{2\pi\Delta l}{\beta}\right) - 2\cosh\left(\frac{2\pi\Delta t}{\beta}\right)}\right) \tag{4.98}$$

We basically got the same result of entanglement entropy for the time-dependent case of a system at finite temperature except by some constant that can be ignored compared to the large value of the logarithm term. We now include a graph in order to visualize the geodesic trajectory in the Lorentzian BTZ black hole (see figure4.5). We can see that extending the time $t \in \mathbb{R}$ we have a infinite cylinder of radius r_c,

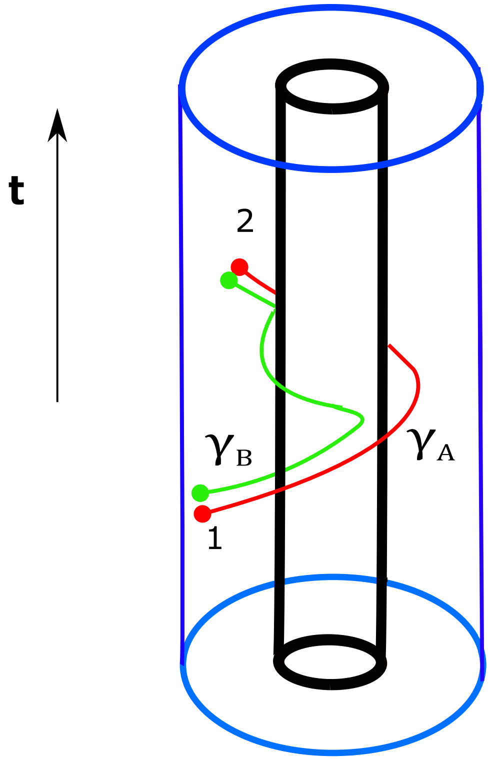

Figure 4.5: Geodesic path in the non-static BTZ black hole corresponding to subsystem A and subsystem B.

but it has a event horizon of the BTZ black hole of radius r_+. The black cylinder represents the BTZ black hole, and we denoted γ_A and γ_B as the geodesics paths corresponding to subsystem A and subsystem B respectively. This graph illustrates that both geodesic paths are different, we can see that for a larger size of subsystem A than its complement subsystem B, γ_A surrounds the black hole, while γ_B doesn't. So we can finally say that when a system is at finite temperature, this is dual to a BTZ black hole in the gravity side, which it will describe a quantum mixture state, which breaks the equality of entanglement entropies of our two subsystem A and B.

Chapter 5

Summary and conclusions

This chapter will present a brief summary of the two computational procedures used in this thesis to obtain entanglement entropy finalizing some conclusions about the results that were obtained.

I explained in Chapter 2 that the way of computing entanglement entropy for a certain quantum system relies in dividing its Hilbert space $\mathcal{H}$ into two subsystems A and B. This was possible by assuming that each subsystem has their own Hilbert space well defined where I denoted them as $\mathcal{H}_A$ and $\mathcal{H}_B$, and they must satisfy that their direct product yields the total Hilbert space of the system ($\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_B$).

The quantum states $|\Psi\rangle$ can be used in order to construct the density matrix ρ, which in general it could be time-dependent if and only if the quantum states evolve with time. We have seen that this density matrix is the key for computing entanglement entropy, which in general one uses only the reduced version of this matrix respect to one of both subsystems A or B. The study of this is important since entanglement entropy is a good measurement of how entangled or quantum a wave function is. The fact that one only compute entanglement entropy for one subsystem, which in this case I chose A, is equivalently saying that S_A is the entropy for an observer who is only accessible to subsystem A and cannot receive any information from B. This subsystem B would be equivalent to the inside of a black hole after crossing the event horizon for an observer who is outside of the horizon. That is why the holographic entanglement entropy satisfy an area law similar to the Bekenstein Hawking entropy [6].

Entanglement entropy is well defined in a quantum field theory, which in this project we worked in the quantum field theories that preserves angles (CFT). We have seen that the conformal field theory satisfies some symmetries which simplifies the calculation of correlation functions of a certain operator. We have seen that entanglement entropy in the CFT is hard to compute if one tries using the quantum path integral formulation of the density matrix (2.19). This is why in the 2-dimensional CFT it is always used the Replica Trick, which consists in replicating the system and then passing the replicated world sheets into the target space recovering locality. Doing this, the interval of the subsystem being studied defines a branch cut to the world sheet, and in order to compute the Rényi entropy ,which is related to Von Neumann entropy, one has to glue all this world sheets together by some local fields called twist fields.

I worked in the CFT using three configurations of a quantum system by having the quantum states evolving in time (covariant case): the infinite long system at zero temperature, the finite size case at criticality and the last one was a infinite long system at a finite temperature. These computations depends of the conformal transformations being used in the CFT, and the computations for the two point functions of the twist fields were easy to solved, this is because of the symmetries involved in the CFT.

The results of entanglement entropy for the infinite long and finite size at zero temperature case yields the same result if one considers the size l of subsystem A to be small in comparison with the size L of the total system. Also I observed that for these two configurations, the entanglement entropy corresponding to both subsystems A and B are equal ($S_A = S_B$). This tells us that we have only pure states when the system is at zero temperature.

However, I observed that this fact totally changes when the quantum system is at some finite temperature β. The result of entropy I got in this case, means that the equality for both sybsystems (2.4) doesn't hold anymore. This tells us that the density matrix is in a mixed state generically.

In the other hand, I used the holographic method to calculate entanglement entropy. As I previously explained, the holographic method is another example of the AdS/CFT correspondence, which claims that the (d+1)-dimensional conformal field theory (CFT_{d+1}) is equivalent to the supergravity on the (d+2)-dimensional anti-de-Sitter space (AdS_{d+2}), where is expected that the CFT_{d+1} sits in the boundary of the AdS_{d+2}. In this case I only worked with the AdS_3/CFT_2 correspondence.

I explained that for the covariant prescription of the holographic method consists in extremizing the length of the interval between two points in a specific metric of space-time. In this project I worked with three different metrics which were: Poincaré's, AdS_3 written in global coordinates and the BTZ black hole.

Using the AdS_3 metric, we obtained the same result of entanglement entropy for the finite size length in the CFT, which I observed that using the Poincaré's patch, one obtains the case when the length of subsystem A is too small compared to the total length of the system. Analyzing the geodesics in these metrics, we concluded that those corresponding to subsystem A are equal to those corresponding to B, the physical meaning of this is that the entanglement entropies from our both subsystems A and B are equal, which we already know that in the CFT this means that the system is composed of only pure quantum states.

However when I analysed the covariant prescription of holographic entanglement entropy in the BTZ black hole, based on the time-dependent result of entanglement entropy, I conclude that a quantum infinite long system is at a finite temperature in the CFT this is dual to a BTZ black hole on the gravity side. The geodesics corresponding to both subsystems A and B for this space-time differs from each other, which this means in the CFT side that the system is composed of mixture states.

To conclude we can say that it resulted successful the computations of time-dependent entanglement entropies by using holography since all the results from the CFT side are equal from those obtained in the gravity side. I can say that for the three metrics we used we can always fixed time yielding the known static results discussed in [12], where is not necessarily true for a generic metric of space-time.

Appendix A

Useful Relations in the CFT

This appendix is a brief summary of CFT explained in [18]. We review the minimal theoretical background material in order to understand the calculations of entanglement entropy in the (1+1)d-CFT and also some notations used for the Replica Trick. Also I will explain the definitions and conditions that must be satisfied for conformal invariance and what are their implications.

A.1 Conformal Field Theory

In general Conformal Field Theory (CFT) differs from Quantum Field Theory (QFT). In QFT it is required to start with the classical action of the quantized fields. If we want to compute the correlation functions it is necessary to work with quantum path integrals. Most of the cases the quantum path integrals are difficult and tedious to compute. On the contrary, CFT simplifies this by exploiting its symmetries. Especially this is possible when dealing with 2 dimensions because the algebra of infinitesimal conformal transformations is infinite dimensional.

A.1.1 Conformal Group

Conformal transformations are basically transformations of a plane in d dimensions which locally preserve the angles between any two lines. The conformal transformation is defined by considering differentiable maps: $\phi : U \to V$ where M is a smooth manifold, $U \subset M$ and $V \subset M'$ are open subsets. The map ϕ is called a conformal transformation, if the metric tensor satisfies $\phi^* g' = \Lambda \phi$. Therefore, using a flat space and having $M' = M$ with a constant metric of the form $\eta_{\mu\nu} = diag(-1, ..., +1, ...)$:

$$\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\eta_{\mu\nu} \tag{A.1}$$

Where the scale factor $\Lambda(x) = 1$ corresponds to the Poincaré group which consists of Lorentz transformations.

The goal of this thesis is to study the holography of entanglement entropy using AdS_3/CFT_2, this means that in the CFT the biggest concern is to study the conformal group in two dimensions. Let's analyse first the conditions that must be satisfied in order to get a conformal invariance in d dimensions. By doing an infinitesimal coordinate transformation in first order of the small parameter $\epsilon(x)$ we get:

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + O(\epsilon^2) \tag{A.2}$$

Using this infinitesimal coordinate transformation we can insert this in relation (A.1), which gives:

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu} \tag{A.3}$$

Where $\partial.\epsilon = \partial^{\mu}\epsilon_{\mu}$, this is important because if we use the infinitesimal transformation (A.2) and the conformal transformation from (A.1) we now get an expression for the the scalar factor $\Lambda(x)$:

$$\Lambda(x) = 1 + \frac{2}{d}(\partial.\epsilon) \tag{A.4}$$

If we restrict the CFT in two dimensions (d=2) and we use equation from (A.3) we basically get the following differential equations:

$$\partial_0\epsilon_0 = \partial_1\epsilon_1, \qquad \partial_0\epsilon_1 = -\partial_1\epsilon_0 \tag{A.5}$$

These are the Cauchy-Riemann equations, so it is straightforward to think that we can use complex variables in the following way:

$$z = x^0 + ix^1, \qquad \epsilon = \epsilon^0 + i\epsilon^1, \qquad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \tag{A.6}$$

In order to get the corresponding equations of $\bar{z}, \bar{\epsilon}$, we simply use the complex conjugate of each equation from (A.6). Using these results we simply observe that the metric tensor transforms under $z \to f(z)$ by the following form:

$$ds^2 = dzd\bar{z} \to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}dzd\bar{z} \tag{A.7}$$

Where we recognize that $|\frac{\partial f}{\partial z}|^2$ is the scalar factor. For a conformal transformation in two dimensions the function $\epsilon(z)$ is holomorphic in some open set because it satisfies Cauchy-Riemann equations. It can be considered that this function has isolated singularities outside this open set. Thus, $\epsilon(z)$ can be expanded by Laurent series around $z = 0$, so basically this will be of the form: $\epsilon(z) = -\sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}$. Where ϵ_n is a constant. The generators corresponding to a transformation for a particular n are:

$$l_n = -z^{n+1}\partial_z \tag{A.8}$$
$$\bar{l_n} = -\bar{z}^{n+1}\partial_z \tag{A.9}$$

Relation (A.8) is the generator related to the functions $\epsilon(z)$ and (A.9) is the one related to the complex conjugate $\bar{\epsilon}$. These two are independent copies of the Witt algebra, we can use z and $\hat{z}$ as independent variables. This is why is sufficient to express the twist fields as $\mathcal{T}_n(z, \hat{z})$ in chapter 2.

Since $n \in \mathbb{Z}$ we can notice that the number of infinitesimal conformal transformation is infinite. These generators must determine a corresponding algebra, therefore these generators must be related by the following commutators:

$$[l_m, l_n] = (m-n)l_{m+n} \tag{A.10}$$
$$\left[\bar{l}_m, \bar{l}_n\right] = (m-n)\bar{l}_{m+n} \tag{A.11}$$
$$\left[l_m, \bar{l}_n\right] = 0 \tag{A.12}$$

The first commutator represent one copy of the Witt algebra, the other two relations implies that there is another copy that commutes with the first one. Thus, according to [18], we can say that the algebra of an infinitesimal conformal transformation in an Euclidean two-dimensional space is infinite dimensional.

We can notice that on the Euclidean plane $\mathbb{R}^2 \simeq \mathbb{C}$, the generators are l_n are not everywhere defined. There is a problem at $z = 0$, so we can't work only with the complex plane $\mathbb{C}$. Also on the Riemann sphere $S^2 \simeq \mathbb{C} \cup [\infty]$, which is the conformal compactification of $\mathbb{R}^2$, not all of the generators are well defined. For z=0 we find that l_n is non singular only when $n \geq -1$. The other problem would be if $z \to \infty$, since taking this limit and changing the coordinates to $z = -\frac{1}{w}$, there is a non-singular point at $w = 0$ only for $n \leq 1$. Therefore, the globally defined transformations on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$ are generated by $[l_{-1}, l_0, l_1]$.

We can observe that the operator l_{-1} is the generator of translations $z \to z + b$, the operator l_0 generates transformations $z \to az$ with $a \in \mathbb{C}$, and l_1 generates the known Special Conformal Transformations, which are translations for the variable $w = -\frac{1}{z}$. We can conclude from this that the operators l_{-1}, l_0, l_1 are the generators corresponding to the the following transformation:

$$z \to \frac{az + b}{cz + d} \tag{A.13}$$

Where $a, b, c, d \in \mathbb{C}$ and since $ad - bc$ must be different from 0, one can always scale these constants in order to have $ad - bc = 1$.These conditions make us conclude that the conformal group of the Riemann sphere is the Möbius group $SL(2, \mathbb{C})/\mathbb{Z}_2$.

This form of conformal mapping (A.13) is used when solving entanglement entropy for the real line case.

It is important to quote that the Witt algebra of infinitesimal conformal transformations admits a central extension. The elements corresponding to central extension of the Witt algebra are L_n with $n \in \mathbb{Z}$. This central extension satisfies the following commutator equation:

$$[L_m, L_n] = (m-n)L_{m+n} + cp(m,n) \tag{A.14}$$

Where $p : \mathbb{Z}X\mathbb{Z} \to \mathbb{C}$ is bilinear and c is what is called as the CFT charge. An expression for the bilinear p(m,n) can be found by analysing the antisymmetry of the commutator, and one can modify the generators in order that the commutators doesn't depend of bilinear $p(m,n)$ (see calculation in [blumenhagen2009introduction]). In the end the commutator that describes the Virasoro algebra Vir_c with central charge c is:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{A.15}$$

A.1.2 Primary Fields

Working in a 2-dimensional, conformal field theory, specifically in Euclidean space, as we mentioned before, we can identify $\mathbb{R}^2 \simeq \mathbb{C}$, by introducing the complex variables as we did in (2.6). Since the generators of Witt algebras are expressed in terms of z and $\bar{z}$, these two complex variables can be treated as independent terms. For the fields ϕ in CFT, we practically have $\mathbb{R}^2 \to \mathbb{C}^2$, which means:

$$\phi(x_0, x_1) \to \phi(z, \bar{z}) \tag{A.16}$$

Where $x_0, x_1 \in \mathbb{R}^2$ and $z, \bar{z} \in \mathbb{C}^2$. The fields which depend of z is called chiral fields $\phi(z)$ and $\phi(\bar{z})$ are the anti-chiral fields. The chiral field is called a primary field of conformal dimension $(h, \bar{h})$ when it transforms under conformal transformations $z \to f(z)$ according to:

$$\phi(z, \bar{z}) \to \phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(z, \bar{z}) \tag{A.17}$$

If we want to study how the primary fields behaves under infinitesimal conformal transformations, one can take the map $f(z) = z + \epsilon(z)$ having $\epsilon << 1$. Under these transformations one get that the transformation of a primary field under infinitesimal conformal transformations one have:

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z, \bar{z}) = (h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\phi(z, \bar{z}) \tag{A.18}$$

A.1.3 Energy-Momentum Tensor

In general, for any field theory it is defined in terms of a Lagrangian action, which it basically has important properties of the theory. The energy-momentum tensor is one of them, which can be deduced by varying the action with respect of the metric. Since the algebra of infinitesimal conformal transformations in 2 dimensions is infinite dimensional, there are strong constraints on a CFT. The CFT is a powerful theory, that we don't even need to have the explicit form of the action.

We can recall Noether's theorem since it basically claims that for every continuous symmetry in a Field theory, there is a current j^μ that it will be conserved. In the CFT, what it looks for is the conformal symmetry $x^\mu \to x^\mu + \epsilon^\mu$. The conserved current associated with the conformal symmetry is:

$$j_\mu = T_{\mu\nu}\epsilon^\nu \tag{A.19}$$

Where the $T_{\mu\nu}$ is symmetric and is the energy-momentum tensor. Since it must be a conserved current, using the fact $\partial_\mu j^\mu = 0$, one can prove that the energy-momentum tensors are traceless in the CFT.

$$T^\mu_\mu = 0 \tag{A.20}$$

For this conserved current, there is a conserved charge which is related to the j^0 in the following form:

$$Q = \int dx^1 j_0 \tag{A.21}$$

The integral above is having $x^0 = constant$. In the field theory, this conserved charge is actually the generator of symmetry transformations for an arbitrary operator A:

$$\delta A = [Q, A] \tag{A.22}$$

The commutator above is evaluated at equal times. Changing variable $z = e^{x_0}e^{ix_1}$, it basically represents the mapping from an infinite cylinder to the complex plane. Since we have the x_0 fixed this basically means that $|z| = constant$ too, so the integral that defines the conserved charge is transformed into a contour integral $\oint dz$ and one can generalize (2.21) in terms of contour integrals:

$$Q = \frac{1}{2\pi i}\oint \left(dzT(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\right) \tag{A.23}$$

This relation is useful in order to calculate the variation $\delta\phi$, using (2.22). So we practically have:

$$\delta_{\epsilon\bar{\epsilon}}\phi(z,\bar{z}) = \frac{1}{2\pi i}\oint_C dz[T(z)\epsilon(z),\phi(w,\bar{w})] + \frac{1}{2\pi i}\oint_C d\bar{z}[\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}),\phi(w,\bar{w})] \quad (A.24)$$

The problem with the expression above (A.21) is that it has an ambiguity since one has to decide whether w or $\bar{w}$ is inside or outside of the contour C. In QFT, it is common that the correlation functions are only defined in terms of time ordered products. Considering our infinite cylinder mapping $z = e^{x_0}e^{ix_1}$, the time ordering becomes a radial ordering, so the product of two observables $A(z)B(w)$ makes sense for $|z| > |w|$. In that sense if we compute the contour path integral we will have:

$$\oint dz[A(z),B(w)] = \oint_{|z|>|w|} dzA(z)B(w) - \oint_{|z|<|w|} dzB(w)A(z) \quad (A.25)$$

$$= \oint_{C(w)} dzR(A(z)B(w)) \quad (A.26)$$

Where $R(A(z)B(w))$ is the radial ordering of two operators. In that sense the variation of the primary field will be:

$$\delta_{\epsilon\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{C(w)} dz\epsilon(z)R(T(z)\phi(w,\bar{w})) + antichiral \quad (A.27)$$

Comparing relation (A.27) with (A.24) we can obtain:

$$R(T(z)\phi(w,\bar{w})) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + ... \quad (A.28)$$

We can rewrite (2.18) in terms of contour integrals:

$$h(\partial_w\epsilon(w))\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{C(w)} dz\frac{h\epsilon(z)}{(z-w)^2}\phi(w,\bar{w}) \quad (A.29)$$

$$\epsilon(w)(\partial_w\phi(w,\bar{w})) = \frac{1}{2\pi i}\oint_{C(w)} dz\frac{\epsilon(z)}{z-w}\partial_w\phi(w,\bar{w}) \quad (A.30)$$

The field $\phi(z,\bar{z})$ is called primary with conformal dimensions $(h,\bar{h})$, if the operator product expansion, between the energy-momentum tensors and $\phi(z,\bar{z})$:

$$T(z)\phi(w,\bar{w}) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) \quad (A.31)$$

$$\bar{T}(\bar{z})\phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) \quad (A.32)$$

Now, let's study the behaviour of the energy-momentum tensor under conformal transformations. To be precise, when there is a conformal transformation $f(z)$, the energy-momentum tensor behaves as:

$$\boxed{T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T\left(f(z)\right) + \frac{c}{12} S\left(f(z), z\right)} \tag{A.33}$$

Where $S(w, z)$ is what is known Schwarzian derivative:

$$S(w, z) = \frac{1}{(\partial_z w)^2}\left((\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2\right) \tag{A.34}$$

We will need this in our computations, because we will always make a change of coordinates and then take the expectation value of the energy-momentum tensor, since as we shall explain later, it is related to the n-point correlation function. In QFT, the correlation function can be computed with perturbative approach via canonical quantization or path integral method. None of these will be used when calculating the 2-point function for the fields in the CFT, in fact it is described by the symmetries.

The last thing to talk about the energy-momentum is the Ward Identity, This is related to the calculation of n-point functions. Basically by using the primary fields ϕ_i and relations (A.29) and (A.30)one derives the Ward Identity:

$$\begin{aligned}\langle T(z)\phi_1(w_1, \bar{w}_1)...\phi_N(w_N, \bar{w}_N)\rangle =& \sum_{i=1}^{N}\left(\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\partial_{w_i}\right)\times \\ &\times \langle\phi_1(w_1, \bar{w}_1)...\phi_N(w_N, \bar{w}_N)\rangle\end{aligned} \tag{A.35}$$

In our case we only need the 2-point functions, so we must consider the invariance under translations $f(z) \to z + a$ which is generated by L_{-1} and invariance under L_0 that corresponds to rescaling of the form $f(z) = \lambda z$. If we have:

$$\langle\phi_1(z)\phi_2(w)\rangle = g(z, w) \tag{A.36}$$

For the translation invariance, it actually implies that $g(z, w) = g(z - w)$ and the rescaling invariance yields the condition $\lambda^1\lambda^2 g(\lambda(z-w)) = g(z-w)$, so we get:

$$\langle\phi_1(z)\phi_2(w)\rangle = \frac{d_{12}}{(z-w)^{h_1+h_2}} \tag{A.37}$$

Where d_{12} is the structure constant. The last invariance that it must satisfy is from generator L_1, in other words the invariance under transformations of the form $f(z) \to -1/z$. This last invariance can only by satisfied if $h_1 = h_2$. Therefore the $SL(2, \mathbb{C})/\mathbb{Z}_2$ conformal symmetry fixes the two-point function of two chiral semi-primary fields to be:

$$\boxed{\langle \phi_i(z)\phi_j(w)\rangle = \frac{d_{ij}\delta_{h_i,h_j}}{(z-w)^{2h_i}}} \tag{A.38}$$

This is the most important result that we need in order to calculate the entanglement entropy using the Replica Trick. The problem will be in calculating the conformal dimensions h, so basically we will need to use Ward Identity for that.

Bibliography

[1] V. E. Hubeny, M. Rangamani and T. Takayanagi, *A covariant holographic entanglement entropy proposal*, Journal of High Energy Physics **2007**, 062 (2007).

[2] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory*Advanced book program (Westview Press Reading (Mass.), Boulder (Colo.), 1995), Autre tirage : 1997.

[3] G. Hooft, *Dimensional reduction in quantum gravity*, arXiv preprint gr-qc/9310026 (1993).

[4] L. Susskind, *The world as a hologram*, Journal of Mathematical Physics **36**, 6377 (1995).

[5] B. de Wit, *BPS black holes*, Nuclear Physics B-Proceedings Supplements **171**, 16 (2007).

[6] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, Physics Letters B **379**, 99 (1996).

[7] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, International journal of theoretical physics **38**, 1113 (1999).

[8] J. Polchinsky*String Theory*, Vol. II (Cambridge University Press, 2005).

[9] M. Caraglio and F. Gliozzi, *Entanglement entropy and twist fields*, Journal of High Energy Physics **2008**, 076 (2008).

[10] J. Cardy, O. A. Castro-Alvaredo and B. Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, Journal of Statistical Physics **130**, 129 (2008).

[11] P. Calabrese and J. Cardy, *Entanglement entropy and conformal field theory*, Journal of Physics A: Mathematical and Theoretical **42**, 504005 (2009).

[12] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence*, Physical review letters **96**, 181602 (2006).

[13] G. de Berredo-Peixoto and M. Katanaev, *Inside the BTZ black hole*, Physical Review D **75**, 024004 (2007).

[14] S. Ryu and T. Takayanagi, *Aspects of holographic entanglement entropy*, Journal of High Energy Physics **2006**, 045 (2006).

[15] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge university press, 2010).

[16] P. Calabrese and J. Cardy, *Evolution of entanglement entropy in one-dimensional systems*, Journal of Statistical Mechanics: Theory and Experiment **2005**, P04010 (2005).

[17] T. Nishioka, S. Ryu and T. Takayanagi, *Holographic entanglement entropy: an overview*, Journal of Physics A: Mathematical and Theoretical **42**, 504008 (2009).

[18] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory: With Applications to String Theory*Lecture Notes in Physics (, 2009).