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Exact Results in Supersymmetric Field Theories

A dissertation on the defect and deformed

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Abstract
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Quantum field theories (QFTs) are the most precise descriptions of the physical reality that humanity has found. Yet exact predictions are often missing as most computations are notoriously difficult to carry out. One generally resorts to perturbation theory which immediately limits the regime of validity. The need of better computational techniques and a deeper understanding of quantum field theory is evident.

The highly symmetric N=4 SYM theory offers guidance in this quest. The theory's maximal supersymmetry and conformal invariance have allowed for the development of several computational techniques, most notably the AdS/CFT correspondence, supersymmetric localization and applications of integrability. These methods provide the rarity of exact results in a fully interacting QFT and shine light on regimes inaccessible by traditional computations.

The insights drawn from N=4 SYM can be extended into more general settings through deformations and modifications. Three such modifications are the β-deformation, the massive deformation N=2* and N=4 SYM with a defect. This thesis summarizes a number of exact results for these three settings through: *i)* a spin-chain analogy for two-point functions in the defect N=4 SYM, *ii)* a vacuum solution for the β-deformed defect N=4 SYM and its spin-chain interpretation of one-point functions, *iii)* a detailed study of the phase transitions in N=2* applying localization and *iv)* an adaptation of the Quantum Spectral Curve to explicit calculations of anomalous dimensions in β-deformed N=4 SYM.

Keywords: AdS/CFT correspondence, integrability, localization, supersymmetric deformations, Quantum Spectral Curve, defect CFT, one-point functions

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Dedicated to my grand parents

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I E. Widen, "Two-point functions of SU(2)-subsector and length-two operators in dCFT," Phys. Lett. B **773**, 435 (2017) doi:10.1016/j.physletb.2017.08.059 [arXiv:1705.08679 [hep-th]].

II E. Widen, "One-point functions in β-deformed $\mathcal{N} = 4$ SYM with defect," JHEP **1811**, 114 (2018) doi:10.1007/JHEP11(2018)114 [arXiv:1804.09514 [hep-th]].

III J. G. Russo, E. Widen and K. Zarembo, "N= 2* Phase Transitions and Holography," [arXiv:1901.02835 [hep-th]].

IV C. Marboe and E. Widen, "The fate of the Konishi multiplet in the β−deformed Quantum Spectral Curve," [arXiv:1902.01248 [hep-th]].

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Contents

1. Introduction

Quantum field theory (QFT) is one of the most successful frameworks found in science to describe the physical world. When it comes to quantitative predictions it is, in fact, *the* most accurate description known, e.g. the electromagnetic moment of an electron has been predicted and experimentally verified with a precision of twelve digits. Rather than a single theory, it is a versatile framework and the applicability ranges from condensed matter systems, phase transitions to the most fundamental particles. The Standard Model, as it is known, is a QFT description of all the basic building blocks that have so far been observed. It describes all particles as excitations in functions called fields and how they interact through three of the four fundamental forces: the strong, the weak and the electromagnetic force. In this description, an extraordinary amount of physical processes are captured in a very small set of mathematical expressions and the validity has been tested and confirmed by a multitude of experiments. The latest major discovery was made at the Large Hadron Collider at CERN where the predicted Higgs boson was detected, confirming the existence of the final particle within the Standard Model.

The descriptive language of fundamental particles is one of symmetries, here used in the more general sense as a mathematical operation that can be applied to the governing equations such that they stay invariant. Such symmetries are described by group theory and particle theories are classified according to which group of symmetries they possess. A particular class of these theories are gauge theories. The Standard Model belongs to this class and has the gauge group $SU(3) \times SU(2) \times U(1)$.

As successful the Standard Model has been so far, there are still a lot of questions and problems with it. First of all, it is not a complete description of the physical universe as both the forth force, gravity, and an explanation for dark matter, are missing. All attempts to fit the gravitational force within the framework of QFT have yet failed to produce sensible theories (they are non-renormalizable).

A long-time candidate for a quantum description of gravity is string theory. Instead of point-like particles, it assumes one-dimensional objects called strings to be the fundamental constituents. String theory has been around since the 70's but so far not provided a unified predictive theory for all four forces. Nevertheless, it has brought a lot of new insights to theoretical physics and we will return to a more recent discovery that revealed a surprising link between strings and quantum field theory.

Second, the Standard Model is problematic because it is simply difficult to handle. Despite its beautiful formulation, it is practically challenging to

compute physical quantities that can be compared to experiments. Typically, one has to resort to perturbation theory in which the particle interactions are assumed to be small and can be treated as corrections to the calculable results of the interaction-free theory. This is normally done through expansions in the coupling constants that parameterize the interaction strengths.

"Constants" is however a bit of a misnomer as these couplings normally depend on the energy scale at which the physical process occurs. They are said to be running with the energy scale. This means that perturbation calculations only are valid in small regimes. For example, the strong force that governs the interactions among quarks is small only at very large energies (asymptotic freedom) while their behavior at small energies is completely inaccessible using perturbation theory.

This problem is glaring within the Standard Model but is, in fact, a general issue of QFT. It is very hard to understand quantum field theories in full detail. This calls for studies of new and more powerful theoretical tools and hopefully some exact results in special cases that may guide the understanding of QFTs in general.

$\mathcal{N} = 4$ Super-Yang-Mills theory

One very special quantum field theory in which one could hope to develop techniques and obtain exact results is $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory. It functions a bit like a theoretical laboratory as it is significantly simpler to handle while still being a fully-fledged interacting quantum field theory. The simplifications are due to its large set of symmetries. It is the unique four-dimensional theory with maximal supersymmetry, and the name $\mathcal{N} = 4$ indicates the sets of supercharges.

Supersymmetry is a suggested symmetry for particle theories which implies a symmetry between the number of bosons and fermions, the two particle families. So far, it has not been observed in experiments but it has some very attractive theoretical features. Most supersymmetric particle theories are thus not candidates for realistic models but serve as theoretical exploration to deepen the understanding of QFTs.

$\mathcal{N} = 4$ SYM is special among them in that, on top of the maximal supersymmetry, it is what is called conformally invariant; it is a four-dimensional example of a conformal field theory (CFT).

Among the conformal symmetries, there is the symmetry under rescalings. Everything described by a CFT is completely independent of the physical scale. The generator of such rescalings within the field theory framework is the dilatation operator D and its action on other field theory operators depends on their corresponding conformal (or scaling) dimension Δ. Mathematically, the dimension is the eigenvalue under rescalings

$$D\mathcal{O}(0) = \Delta_{\mathcal{O}}\mathcal{O}(0) \tag{1.1}$$

and is classically[1] the mass dimension of the operator $\mathcal{O}$ and therefore half-integers. The quantization of $\mathcal{N} = 4$ SYM, as with most QFTs, requires a process called renormalization in which operators may start to mix with each other. As a consequence, D acts as a matrix and only certain (linear) combinations of operators have well-defined scaling dimensions. These, in turn, may develop a fractional part

$$\Delta = \Delta_0 + \gamma \tag{1.2}$$

where γ is called the anomalous dimension. Finding these anomalous dimensions and the corresponding operator combinations is a central problem in any CFT. They constitute half of what is called the conformal data (the other being the coefficients in the three-point functions) with which, at least in principle, any quantity in the theory can be calculated. Such knowledge would thus give us one of those sought for examples of solvable QFTs.

The AdS/CFT correspondence

$\mathcal{N} = 4$ SYM was found already in 1977 but the interest in it was renewed by a more recent discovery in 1997. In a seminal paper, Maldacena conjectured that there is a strong link between $\mathcal{N} = 4$ SYM and string theory. In fact, he boldly stated that $\mathcal{N} = 4$ SYM is dual[2] to IIB (closed) string theory in a spacetime geometry called $AdS_5 \times S^5$. Dual in this context means that they describe the same physics and are thus equivalent theories, just phrased in different languages. That means that one-dimensional strings in a ten-dimensional curved spacetime would capture the same physics as do point-like operators in a four-dimensional spacetime. It is an example of what is called the holographic principle, where the details of a physical system are described fully in a lower dimensional space. The original formulation is called the AdS/CFT correspondence but several examples of other holographic dualities have since been found.

This new avenue of research spawned a flurry of activity and the past two decades have been filled with a multitude of non-trivial tests. The conjecture has not yet been proven but after twenty years of withstanding all tests thrown at it, few doubt its validity.

An important aspect of the AdS/CFT correspondence is that it is a so called strong/weak duality. It means that the strongly coupled regime in the field theory, where perturbative methods do not reach, is dual to a weakly coupled regime in string theory, where known calculation techniques *do* work. It is thus possible to obtain field theory results, normally not accessible, by doing a manageable string theory calculation and vice versa.

In order to check and explore the validity of this remarkable correspondence, however, the strong/weak nature of it requires some exact results in the

[1] Field theories that describes physics without anything quantum mechanical are called classical. The process of promoting a field theory to incorporate quantum effects is called quantization.
[2] in the large N limit of an infinitely large rank of the gauge group

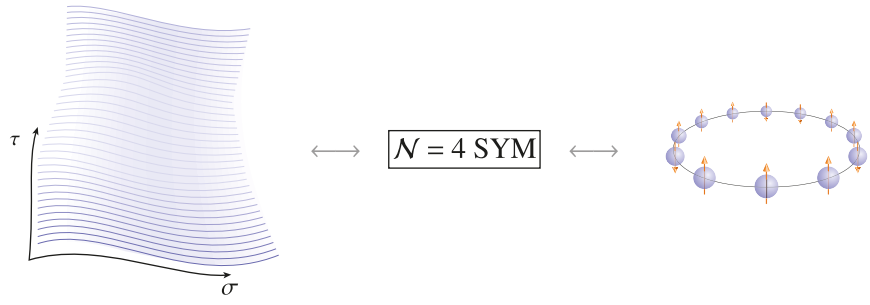


Figure 1.1. The very special theory $\mathcal{N} = 4$ SYM is both dual to string theory via the AdS/CFT correspondence and enjoys an integrable spin-chain interpretation. Both allow for computations of quantities that are inaccessible by other means. The picture illustrates string theory to the left with a string propagating through spacetime and the spin-chain to the right as a periodic lattice with a number of particles that interact with each other through spin-couplings.

strongly coupled regimes. In particular two methods have been used and developed in this process: supersymmetric localization and integrability. Both have stories of their own but we will focus on them in the presented context.

Localization

Supersymmetric localization for $\mathcal{N} = 4$ SYM was first conjectured in 2000 and consecutively proven in 2007. It allows for an immense simplification of the partition function of a range of supersymmetric field theories. The partition function is used to calculate the expectation values of different quantities by integrating over all possible field configurations. Localization utilizes supersymmetry to localize this integration from an infinite-dimensional space onto a finite one and is valid for any value of the coupling constants. This allows for exact results in both the strong, weak and intermediate regimes but comes at a price. Only a small subset of certain supersymmetric quantities are calculable using this technique. Notable examples are Wilson loops, which have been studied extensively, but also the free energy and susceptibility are attainable using localization.

Integrability

The other important method for exact results we encounter in this thesis is that of integrability. In quantum theories, integrability is loosely defined as the existence of an infinite tower of commuting charges, that is to say an infinite number of symmetries. This vast symmetry makes it possible to formulate integrable theories in terms that lay bare their entire structure. Such theories are thus called exactly solvable theories.

The concept of integrability enters in a wide range of systems, e.g. the harmonic oscillator, planetary motion and vertex models, but are yet rare. An important class of models are integrable spin-chains. They are one-dimensional

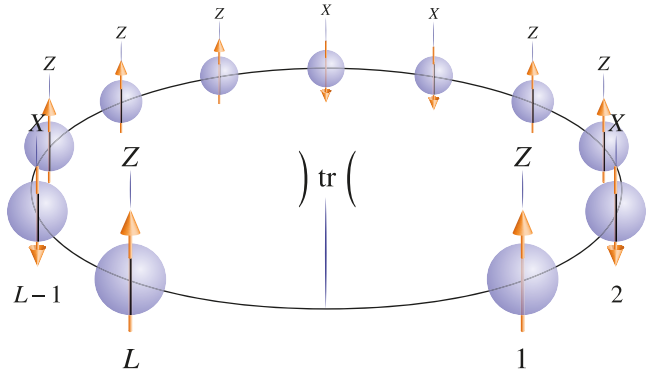


Figure 1.2. A sketch over the spin-chain analogy of the single-trace operator (1.3). Each field maps to a site in the spin-chain where, in this case, Z maps to a spin-up particle and X maps to a spin-down. Since the trace is cyclic, the chain needs to be periodic.

lattice models of particles that interact with their neighbors through spin-spin couplings. Spin-chains were first introduced as simple models for magnetism and were the first encountered examples of quantum integrable systems. In 1931, Hans Bethe managed to exactly solve one of these models, the Heisenberg chain, by an ansatz that since carries his name. The research field has expanded greatly since then and took a new turn when they turned out to be related to some supersymmetric field theories in the early 2000's.

When studying the spectrum of anomalous dimensions in $\mathcal{N} = 4$ SYM, Minahan and Zarembo observed in [1] that the problem was mathematically identical to solving a spin-chain model. Being indeed two very different theories — a conformal field theory in four dimensions on the one hand and a one-dimensional lattice model on the other — the connection is made through composite single-trace operators.

Let us make this more concrete by considering an operator built out of two (of the three complex) scalar fields Z and X in $\mathcal{N} = 4$ SYM. The single-trace operator

$$\mathcal{O} = \mathrm{tr}\,(ZXZZX\cdots ZXZ) \tag{1.3}$$

could be viewed as chain with a site at each scalar where the scalar type Z or X corresponds to either spin-up or spin-down. Since the trace is cyclic, the chain needs to be periodic, like in figure 1.2. The key insight in [1] was that the dilatation operator acts on these operators as does the spin-chain Hamiltonian,

the operator that measures energy. We have the picture

$\mathcal{N}=4$ SYM		Spin-chain
$\mathrm{tr}\,(ZXZZX\cdots ZXZ)$	$\longleftrightarrow$	
D	$\longleftrightarrow$	H
γ	$\longleftrightarrow$	energy levels.

The dilatation eigenvalues, i.e. the anomalous dimensions, are hence described by the energy levels in the spin-chain and the spin-chain energy eigenstates correspond to the operator combinations with definite scaling dimensions.

Through this analogy, a lot of the conformal data is calculable via integrability techniques that have been developed and refined into an extensive machinery over the past 80 years. The encounter of integrability within $\mathcal{N}=4$ SYM inspired to push this development even further.

The first observation in [1] was made only for a part of the theory, the 1-loop $SO(6)$-subsector of scalars, but the principle could be extended to the full set of fields in $\mathcal{N}=4$ SYM. The analogous spin-chain is a $PSU(2,2|4)$ supersymmetric chain, reflecting the full symmetry group of $\mathcal{N}=4$ SYM, and a lot of work has been done to widen the scope of the integrability techniques to these type of spin-chains.

This could have been the whole story but there is, however, a problem with the spin-chain picture. As more perturbative corrections are included in the field theory calculations, accounting for what is called higher loop interactions, the interaction length in the corresponding spin-chain increases. The spin-chain particles begin to talk not only to their neighbors but to their neighbors' neighbors and so forth. When the interaction length is long enough, it starts to wrap around the chain (this happens at loop order L where L is the number of fields in the single-trace operator) and the spin-chain picture breaks down.

Methods have been developed to deal with these complications, such as Lüsher corrections and the thermodynamic Bethe ansatz. However, the state of the art technique to calculate anomalous dimensions in $\mathcal{N}=4$ SYM is the so called Quantum Spectral Curve.

The Quantum Spectral Curve (QSC) is a Riemann-Hilbert problem that captures the full integrable structure of (planar) $\mathcal{N}=4$ SYM in a highly compact way and allows for extremely efficient calculations. Inspired by the eigenvalue spectrum of a certain operator[3] in the dual string theory, it connects string theory, through SYM, all the way to spin-chains in one framework.

It was first developed in [2, 3] by Gromov, Kazakov, Leurent and Volin and has been the source of many extraordinary results. Of particular relevance to

[3] i.e. the trace of Lax operators in the integrable σ-model of the worldsheet

this thesis is the approach developed in [4, 5] to find a leading solution and then extend it through an efficient perturbative algorithm.

Deformations and defects

Truly a wonder of a theory, $\mathcal{N} = 4$ SYM is also rather special. That comes as a disadvantage in the attempts to gain understanding of general QFTs through it. An approach to bypass this disadvantage is to start from $\mathcal{N} = 4$ SYM but then deform it in various ways. If carefully done, it can still keep the applicability of the discussed methods but have stronger resemblance with other theories, thus stretching the insights from $\mathcal{N} = 4$ SYM into more general settings.

There are many such deformations, like the η-, γ- and β-deformations. Two such that enter this thesis and they are the β-deformation and $\mathcal{N} = 2^*$ SYM.

The β-deformation.

The former is a deformation of what is called the R-symmetry of $\mathcal{N} = 4$ SYM. Technically, one introduces a non-commutative Moyal-like product that depends on the R-charges of the fields which deforms the Lagrangian with field dependent phases. In the dual geometry, this amounts to squashing the sphere S^5 a bit.

The β-deformation retains some supersymmetry ($\mathcal{N} = 1$) and conformality and there is a corresponding spin-chain picture in which the boundary conditions of the chain are twisted due to the deformation.

There exists a twisted formulation of the QSC [6]. Paper IV aimed to adapt the methods of [4, 5] and to calculate some explicit examples of anomalous dimension in the β-deformation.

$\mathcal{N} = 2^*$ SYM.

The second casting deformation is $\mathcal{N} = 2^*$ SYM. It is a massive theory so it immediately breaks conformal invariance (the mass introduces a physical scale). It has the same field content as $\mathcal{N} = 4$ SYM but some fields (the $\mathcal{N} = 2$ hypermultiplet with two of the three complex scalars and two of the four fermions) are given a mass. It is not integrable but it is treatable with the methods of localization and has the interesting feature of being a non-conformal theory with a known holographic dual[4].

From the localization results, it has been discovered that $\mathcal{N} = 2^*$ SYM has an intricate pattern of phase transitions, very different from regular $\mathcal{N} = 4$ SYM. Paper III probed deeper into the nature of these phase transitions in the strong-coupling regime and showed that they occur in the next-to-next-to-leading order, illuminating why the corresponding phase transitions in holographic dual have not yet been observed.

[4]The dual is the Pilch-Warner background.

A Defect.
A different kind of modification than the deformations is the introduction of a defect in $\mathcal{N} = 4$ SYM. Inspired by holography[5], one can put a (codimensional-one) defect at the coordinate value $z = 0$. It breaks the four-dimensional conformal symmetry but preserves dilatations, thus keeping the scaling dimensions Δ as central objects.

An interesting effect of the defect is that non-trivial one-point functions, normally forbidden by the conformal symmetry, are allowed. They still have an integrable spin-chain interpretation and the defect corresponds to a certain Matrix Product State (MPS) in the spin-chain. A series of papers have studied the one-point functions in terms of spin-chain overlaps with the MPS and have resulted in some neat determinant formulas. It was first discovered for the $SU(2)$-subsector illustrated above, and then consecutively extended into the full scalar sector.[6]

The study of one-point functions in the defect $\mathcal{N} = 4$ SYM was taken into loop calculations in [10, 11]. Building on that work, Paper I studied a special case of two-point functions that could be reduced and written solely in the spin-chain language of the one-point functions.

All these were results for defect $\mathcal{N} = 4$ SYM. Paper II initiated the exploration of the combination of a defect and the β-deformation. It found a vacuum solution admitting a defect and the corresponding MPS in the spin-chain picture. The simplest types of one-point functions were calculated but no simplifying determinant formula was found, as it was shown that the immediate integrable structure of the MPS is broken by the β-twist.

1.1 Thesis outline

This thesis is organized as a introductory summary to help readers follow the appended papers. It naturally has overlaps but also tries to flesh out the underlying concepts in more detail. It is organized in the following six chapters

2. $\mathcal{N} = 4$ **SYM** introduces the foundational theory with its symmetries and the group theoretical representations used in the thesis. It briefly reviews correlation functions in CFT and shows the link between the renormalized theory and the spin-chain picture. It is essential to all four appended papers. The final section introduces the minimal amount of holography in terms of the AdS/CFT required to follow the discussions.

3. Deformations of $\mathcal{N} = 4$ **SYM** specifies the two deformations of $\mathcal{N} = 4$ SYM treated in Papers II-IV: the β-deformation and the $\mathcal{N} = 2^*$ SYM.

4. Localization and matrix models conceptually introduces the techniques of localization and the resulting matrix models from applying localiza-

[5] It is the holographic dual is the D3-D5-brane construction of [7].
[6] Other defects (dual to the D3-D7-brane set-up) have also been considered. See e.g [8, 9].

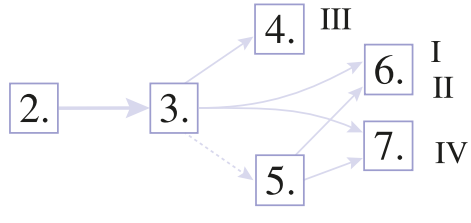
tion to $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2^*$ SYM. It provides the theoretical background for Paper III.

5. Spin-chains and integrability contains fairly self-contained review of the relevant spin-chain models and three formalisms to solve them: the coordinate Bethe ansatz, the algebraic Bethe ansatz and Q-functions and Q-operators. It also discusses the special spin-chain state appearing in the following chapter. It is highly relevant for Paper I and II and helpful to understand Paper IV.

6. $\mathcal{N} = 4$ SYM with a defect discusses how to put a specific defect into $\mathcal{N} = 4$ SYM and the implications for the symmetries, correlation functions and the existence of a vacuum solution. Then the results for one-point functions are reviewed, with focus on the $SU(2)$-subsector and the properties of the corresponding MPS. The scalar propagators in the presence of the defect are reviewed to provide background for the summarized results of Paper I. Lastly, the β-deformation is introduced into the defect theory and the consequences, worked out in Paper II, are stated.

7. Quantum Spectral Curve for the β-deformation provides a crash-course review of the QSC and, in particular, to the adaptations made in Paper IV. The $\mathbf{P}\mu$-system is introduced, as well as an approach to finding the leading solution and how to build on that in a perturbative algorithm.

The interdependencies of the chapters and connections to the papers can be illustrated as:



1.1.1 A note to the appended papers

All four papers are reprinted in greyscale in the second half of this thesis. Since some of them (in particular Paper IV and the plots of Paper II) make heavy use of coloration, the interested reader is encouraged to find the original versions online.

2. $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

$\mathcal{N} = 4$ SYM was first discovered in 1977 in [12] and attracted much interest already from the start. The advent of the AdS/CFT correspondence in 1997 sparked another wave of research into the properties of this theory. As it underlies all the results for this thesis, we begin with reviewing its action, symmetries, conformal properties and role in the AdS/CFT correspondence.

2.1 The action

It is possible to obtain the action of $\mathcal{N} = 4$ SYM in more than one way. The original formulation in [12] arrived at it through a dimensional reduction of $\mathcal{N} = 1$ SYM in 10D. We will first follow that approach but later on also present a construction from the action in $\mathcal{N} = 1$ superspace.

2.1.1 Dimensional reduction from 10D $\mathcal{N} = 1$ SYM

Let us, as stated, first obtain the action of $\mathcal{N} = 4$ SYM by starting from the unique $\mathcal{N} = 1$ SYM action in 10 dimensions and dimensionally reduce it down to 4 dimensions.

The starting point is the ten-dimensional SYM action with a massless spin-$\frac{1}{2}$ Majorana-Weyl fermion Ψ in flat Minkowski space,[1]

$$S_{\text{10D}} = \frac{1}{g^2_{\text{YM}}} \int \mathrm{d}^{10}x \ \mathrm{tr}\left(-\frac{1}{2} F_{MN}F^{MN} + \bar{\Psi}\Gamma^M D_M \Psi\right). \tag{2.1}$$

The field strength F_{MN}, $M,N = 0,1,\ldots,9$, and the covariant derivative are as usual defined in terms of the gauge field A_M ,

$$F_{MN} = \partial_M A_N - \partial_N A_M - \mathrm{i}[A_M, A_N] \tag{2.2}$$

$$D_M = \partial_M - \mathrm{i}[A_M, \cdot\,]., \tag{2.3}$$

and all fields are in the adjoint representation of the gauge group $U(N)$ or $SU(N)$ such that, for instance,

$$A_M = A^a_M T^a, \qquad \mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \qquad a,b = \begin{cases} 0,1,\ldots,N^2-1 & \text{for } U(N), \\ 1,2,\ldots,N^2-1 & \text{for } SU(N), \end{cases} \tag{2.4}$$

[1] Here and throughout the thesis, Einstein's summation convention of repeated indices is understood, except where otherwise stated.

with T^a being the generators of $\mathfrak{u}(n)$ or $\mathfrak{su}(N)$, respectively. Γ^M are the ten-dimensional gamma-matrices.

This action enjoys an $\mathcal{N} = 1$ supersymmetry in form of the transformation

$$\delta_\varepsilon A_M = -\mathrm{i}\bar{\varepsilon}\Gamma_M\Psi, \tag{2.5}$$

$$\delta_\varepsilon \Psi = \frac{\mathrm{i}}{2} F_{MN}\Gamma^{MN}\varepsilon, \tag{2.6}$$

where $\Gamma^{MN} = \frac{\mathrm{i}}{2}\left(\Gamma^M\Gamma^N - \Gamma^N\Gamma^M\right)$ and ε is a constant Majorana-Weyl spinor. This symmetry is the reason for the large symmetry group we will obtain after the dimensional reduction.

Dimensional reduction

We now dimensionally reduce the action (2.1) down to four dimensions. That results in a split of the ten-dimensional gauge field into the four-dimensional gauge field A_μ, with $\mu = 0,1,2,3$, and six real scalars φ_i, $i = 1,2,\ldots,6$ coming from the compactified dimensions. The ten-dimensional fermion undergoes a similar split into four four-dimensional Majorana fermions ψ. The resulting action reads

$$S = \frac{2}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\ \mathrm{tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\varphi_i D^\mu\varphi_i + \frac{\mathrm{i}}{2}\bar{\psi}\Gamma^\mu D_\mu\psi \\ + \frac{1}{2}\bar{\psi}\tilde{\Gamma}^i[\varphi_i,\psi] + \frac{1}{4}[\varphi_i,\varphi_j][\varphi_i,\varphi_j]\Big), \tag{2.7}$$

where again the field strength and covariant derivative are defined as above, but with $\mu,\nu = 0,1,2,3$. The reduction of Γ^M to Γ^μ and $\tilde{\Gamma}^i$ can be found explicitly in [11] but we will not need it for this thesis.

Notation of scalars

The six real scalars φ_i are in the six-dimensional vector representation of the $SO(6)$ R-symmetry, discussed below. It is, however, common to map them to the anti-symmetric representation of $SU(4) \simeq SO(6)$ through the six-dimensional sigma matrices $\Phi_{ab} = -\Phi_{ba} = \varphi_i\Sigma^i_{ab}$. Moreover, a third notation of the now complex scalars is to write them as $\phi^i = \Phi_{i4}$ and $\bar{\phi}_i = (\phi^i)^*$ with $i = 1,2,3$. These transform in the fundamental and anti-fundamental representations of the $SU(3)$-subgroup of $SU(4)$. In this thesis, we will use all three notations with

$$\begin{aligned} Z &= \phi^1 = \varphi_1 + \mathrm{i}\varphi_4\,, \quad X = \phi^2 = \varphi_2 + \mathrm{i}\varphi_5\,, \quad Y = \phi^3 = \varphi_3 + \mathrm{i}\varphi_6\,, \\ \bar{Z} &= \bar{\phi}_1 = \varphi_1 - \mathrm{i}\varphi_4\,, \quad \bar{X} = \bar{\phi}_2 = \varphi_2 - \mathrm{i}\varphi_5\,, \quad \bar{Y} = \bar{\phi}_3 = \varphi_3 - \mathrm{i}\varphi_6\,. \end{aligned} \tag{2.8}$$

Note that the notation and labeling of the scalars differs somewhat between the appended papers.

2.1.2 Conformality, large-N and the planar limit

The action (2.7) of $\mathcal{N} = 4$ SYM has two parameters: the Yang-Mills coupling g_{YM} and the rank of the gauge group N. Most commonly in field theories though, renormalization introduces a mass scale Λ with which the coupling constants run and are hence not freely tunable. However, $\mathcal{N} = 4$ SYM belongs to the small subset of theories for which the couplings are independent of the renormalization scale, i.e. its single β-function satisfies

$$\beta_{g_{\text{YM}}}(\Lambda) = \Lambda \frac{\mathrm{d}g_{\text{YM}}}{\mathrm{d}\Lambda} = 0\,. \tag{2.9}$$

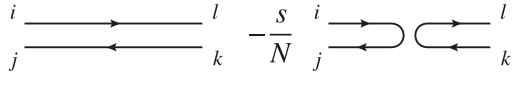
The classical conformal invariance of the action (2.7) is thus preserved perturbatively, and believed to hold also non-perturbatively, such that $\mathcal{N} = 4$ SYM enjoys conformal symmetry at full quantum level [13–18]. The Yang-Mills coupling g_{YM} is hence an entirely free parameter of the theory.

The second parameter N is also at our disposal. Although the color group of the standard model only has $N = 3$, there has been an enormous interest in theories for which N is very large. The reason for this is the observation, first due to 't Hooft in [19], that the coupling constant g_{YM} and N combine in the large N limit into

$$\lambda = g_{\text{YM}}^2 N\,, \qquad\qquad g^2 = \frac{\lambda}{(4\pi)^2}\,. \tag{2.10}$$

λ is called the 't Hooft coupling and as we will exclusively deal with large N theories it will be the relevant coupling through out the thesis. Here we have also introduced the effective planar coupling constant g which is a convenient rescaling of the 't Hooft coupling.

The combination into λ in the large N limit can be seen from the Feynman diagrams in the double-line notation. In this notation, each color index follows a line where two open ends correspond to a Kronecker δ and connected lines are summed over, e.g.



which is the completeness relation[2] for the gauge group generators ($s = 0$ for $U(N)$ and $s = 1$ for $SU(N)$). Any closed line produces a trace and hence a factor of $\delta^i_i = N$. Counting interaction vertices, propagators and closed lines, the Yang-Mills coupling and N combine into λ while the topology of a double-line graph decides its order in N, as exemplified in figure 2.1. The set of leading graphs are all planar, that is to say that they can be drawn flatly on a sphere.

[2] i.e. the relation

$$\sum_{a=s}^{N^2-1} [T^a]^i{}_j [T^a]^k{}_l = \delta^i_l \delta^k_j - \tfrac{s}{N} \delta^i_j \delta^k_l$$

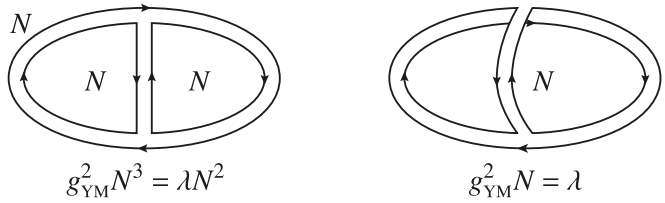


Figure 2.1. An illustration of a planar graph in the double-line notation to the left and a non-planar graph to the right. Notice how the Yang-Mills coupling g_{YM} and the gauge group rank N combine into the 't Hooft coupling λ and how the topology of the graph determines the number of closed lines and hence the order in N.

Consequently, all non-planar graphs are suppressed in the large N limit leaving us only with the small subset of planar graphs, why the limit is also known as the planar limit. The limit is taken in such a way that the relevant expansion parameter λ is kept fixed, hence forcing $g_{\mathrm{YM}} \to 0$. This is a great simplification of the theory, not only because of the vastly reduced number of graphs but also because it admits factorization of correlation functions into their smallest partitions, a process called large N factorization.

Strictly speaking, there are a few caveats to the above statement about planarity. Technically, a graph is called planar if it contributes at the leading order in N if all its open double-lines are contracted at a single vertex at infinity. This will not, however, play a prominent role for this thesis; rather, more details can be found in references [20, 21].

2.1.3 The action from $\mathcal{N} = 1$ superspace

The action for $\mathcal{N} = 4$ SYM can also be obtained from an $\mathcal{N} = 1$ action in superspace. The field content is packaged into one vector superfield V and three chiral superfields Φ^i. The complex scalars ϕ^i are the scalar content of the latter and three of the fermions are their spinors. The fourth fermion is bundled as the gaugino together with the gauge field into the vector superfield. Although the final action for the component fields is equivalent with the action (2.7), this formalism obscures the full $SU(4)$ R-symmetry such that only its $SU(3) \times U(1)$ subgroup is manifest.

The $\mathcal{N} = 1$ superspace action reads

$$S = \mathrm{tr}\Bigg\{ \int \mathrm{d}^4x \Big[\mathrm{d}^4\theta\, \mathrm{e}^{-gV} \Phi_i^\dagger \mathrm{e}^{gV} \Phi^i + \frac{1}{4}\Big(\int \mathrm{d}^2\theta W^\alpha W_\alpha + \mathrm{h.c.} \Big) + \mathrm{i}g \frac{\sqrt{2}}{3!}\Big(\int \mathrm{d}^2\theta\, \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + \int \mathrm{d}^2\bar{\theta}\, \epsilon^{ijk} \Phi_i^\dagger [\Phi_j^\dagger, \Phi_k^\dagger] \Big) \Big] \Bigg\}, \quad (2.11)$$

where the standard superfields in the Wess-Zumino gauge are[3]

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + \mathrm{i}\theta\theta\bar{\theta}\bar{\psi} - \mathrm{i}\bar{\theta}\bar{\theta}\theta\psi + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D, \tag{2.12}$$

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V. \tag{2.13}$$

The derivation operators are

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \tag{2.14}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - \mathrm{i}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \tag{2.15}$$

and the scalar potential is

$$W(\Phi) = \mathrm{i}g\frac{\sqrt{2}}{3!}\epsilon_{ijk}\,\mathrm{tr}\,\Phi^i[\Phi^j,\Phi^k], \tag{2.16}$$

while we temporarily have dropped the subscript of the Yang-Mills coupling $g = g_{\mathrm{YM}}$.

Expanding the action (2.11) into the component fields, performing the integration over the Grassmanian coordinates and substituting the equations of motion for the auxiliary fields brings it to the form equivalent to (2.7).

We will, however, only make very limited use of the superspace formulation. Fuller accounts can for instance be found in [22, 23].

2.2 Symmetries

The $\mathcal{N} = 4$ SYM action has a remarkable amount of symmetry. First, it has the four-dimensional Poincaré symmetry with the six generators $L^\alpha{}_\beta, L^{\dot{\alpha}}{}_{\dot{\beta}}$ of the Lorentz group and the four translations $P_{\alpha\dot{\alpha}} = P_\mu\sigma^\mu_{\alpha\dot{\alpha}}$. Here $\sigma^\mu = \{\mathbb{1}, \sigma^1, \sigma^2, \sigma^3\}$ are the three Pauli matrices collected with the identity $\mathbb{1}$ and map between the Lorentz group $SO(1,3)$ and its $SL(2,\mathbb{C})$ representation. The symmetry generators have the commutation relations

$$[L^\alpha{}_\beta, J_\gamma] = \delta^\alpha_\gamma J_\beta - \tfrac{1}{2}\delta^\alpha_\beta J_\gamma, \qquad [L^\alpha{}_\beta, J^\gamma] = -\delta^\gamma_\beta J^\alpha + \tfrac{1}{2}\delta^\alpha_\beta J^\gamma, \tag{2.17}$$

$$[\dot{L}^{\dot{\alpha}}{}_{\dot{\beta}}, J_{\dot{\gamma}}] = \delta^{\dot{\alpha}}_{\dot{\gamma}} J_{\dot{\beta}} - \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} J_{\dot{\gamma}}, \qquad [\dot{L}^{\dot{\alpha}}{}_{\dot{\beta}}, J^{\dot{\gamma}}] = -\delta^{\dot{\gamma}}_{\dot{\beta}} J^{\dot{\alpha}} + \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} J^{\dot{\gamma}}, \tag{2.18}$$

for any generator J and apply for each of its indices.

[3] We trust the reader not to confuse the auxiliary D-field with the dilatation operator which will use the same symbol below.

This symmetry is extended to the four-dimensional conformal group $SO(2,4) \simeq SU(2,2)$ through the dilatation generator D and the four special conformal generators $K^{\alpha\dot{\alpha}}$. Their commutation relations are

$$[D, P_{\alpha\dot{\alpha}}] = P_{\alpha\dot{\alpha}}\,, \qquad [D, L^{\alpha}{}_{\beta}] = [D, \dot{L}^{\dot{\alpha}}{}_{\dot{\beta}}] = 0\,, \quad [D, K^{\alpha\dot{\alpha}}] = -K^{\alpha\dot{\alpha}}\,, \tag{2.19}$$

$$[K^{\alpha\dot{\alpha}}, P_{\beta\dot{\beta}}] = \delta^{\dot{\alpha}}_{\dot{\beta}} L^{\alpha}{}_{\beta} + \delta^{\alpha}_{\beta}\dot{L}^{\dot{\alpha}}{}_{\dot{\beta}} + \delta^{\alpha}_{\beta}\delta^{\dot{\alpha}}_{\dot{\beta}} D\,. \tag{2.20}$$

The ten-dimensional Lorentz symmetry gives rise to an $SO(6) \simeq SU(4)$ R-symmetry from its compactified dimensions. The gauge field is a singlet under the R-symmetry while the scalars transform in the six-dimensional anti-symmetric representation and the fermions in the (anti-)fundamental representation. The generators commute with any generator with an R-symmetry index according to

$$[R^{a}{}_{b}, J_c] = \delta^{a}_{c} J_b - \tfrac{1}{4}\delta^{a}_{b} J_c\,, \qquad [R^{a}{}_{b}, J^{c}] = -\delta^{c}_{b} J^{a} + \tfrac{1}{4}\delta^{a}_{b} J^{c}\,, \tag{2.21}$$

where again this should be applied to every such index.

Moreover, the theory also has the maximal amount of permitted supersymmetry with $\mathcal{N} = 4$, i.e. 16 supercharges Q^{a}_{α} and $\dot{Q}_{a\dot{\alpha}}$. They commute with the special conformal charges into the 16 superconformal charges S^{α}_{a} and $\dot{S}^{a\dot{\alpha}}$. The (anti-)commutation relations read

$$\{\dot{Q}_{a\dot{\alpha}}, Q^{b}_{\alpha}\} = \delta^{b}_{a} P_{\alpha\dot{\alpha}}\,, \quad [K^{\alpha\dot{\alpha}}, Q^{a}_{\beta}] = \delta^{\alpha}_{\beta}\dot{S}^{a\dot{\alpha}}\,, \quad [K^{\alpha\dot{\alpha}}, \dot{Q}_{a\dot{\beta}}] = \delta^{\dot{\alpha}}_{\dot{\beta}} S^{\alpha}_{a}\,, \tag{2.22}$$

$$\{\dot{S}^{a\dot{\alpha}}, S^{\alpha}_{b}\} = \delta^{a}_{b} K^{\alpha\dot{\alpha}}\,, \quad [S^{\alpha}_{a}, P_{\beta\dot{\beta}}] = \delta^{\alpha}_{\beta}\dot{Q}_{a\dot{\beta}}\,, \quad [\dot{S}^{a\dot{\alpha}}, P_{\beta\dot{\beta}}] = \delta^{\dot{\alpha}}_{\dot{\beta}} Q^{a}_{\beta}\,, \tag{2.23}$$

and

$$\{S^{\alpha}_{a}, Q^{b}_{\beta}\} = \delta^{b}_{a} L^{\alpha}{}_{\beta} + \delta^{\alpha}_{\beta} R^{b}{}_{a} + \tfrac{1}{2}\delta^{\alpha}_{\beta}\delta^{b}_{a} D\,, \tag{2.24}$$

$$\{\dot{S}^{a\dot{\alpha}}, \dot{Q}_{b\dot{\beta}}\} = \delta^{a}_{b}\dot{L}^{\dot{\alpha}}{}_{\dot{\beta}} - \delta^{\dot{\alpha}}_{\dot{\beta}} R^{a}{}_{b} + \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^{a}_{b} D\,. \tag{2.25}$$

The commutation relations with the dilatation operator is of the same form as in (2.19), i.e.

$$[D, J] = \Delta_J J\,, \tag{2.26}$$

where

$$\Delta_Q = \Delta_{\dot{Q}} = +\tfrac{1}{2}\,, \qquad \Delta_S = \Delta_{\dot{S}} = -\tfrac{1}{2} \tag{2.27}$$

$$\Delta_P = +1\,, \qquad \Delta_K = -1\,. \tag{2.28}$$

We repeated the dimensions for $P_{\alpha\dot{\alpha}}$ and $K^{\alpha\dot{\alpha}}$ to emphasize the raising and lowering of the dimension by the two columns of generators.

All in all, these charges generates the full symmetry group $PSU(2,2|4)$.

2.3 Representations of $PSU(2,2|4)$ as a graded super Lie algebra

The spectrum of $\mathcal{N}=4$ SYM naturally organizes itself according to its symmetry group $PSU(2,2|4)$. Let us review some facts about the representation theory in general for $GL(N,M|K)$ and in particular $PSU(2,2|4)$.

Generally, we can label the generators of $\mathfrak{gl}(N,M|K)$ as

$$E_{mn}\,, \qquad m,n=1,2,\ldots,N+M+K\,, \tag{2.29}$$

and introduce two gradings of these according to

$$p_m=\begin{cases}0 & \text{if} \quad m\leq M+N\\ 1 & \text{if} \quad M+N+1\leq m\end{cases} \qquad \text{and} \quad c_m=\begin{cases}1 & \text{if} \quad m\leq N\\ 0 & \text{if} \quad N+1\leq m\end{cases}\,. \tag{2.30}$$

The (anti)-commutation relations with this notation are

$$E_{mn}E_{kl}-(-1)^{(p_m+p_n)(p_k+p_l)}E_{kl}E_{mn}=\delta_{nk}E_{ml}-(-1)^{(p_m+p_n)(p_k+p_l)}\delta_{ml}E_{kn}\,. \tag{2.31}$$

The Cartan subalgebra consists of the generators E_{mm} and have the weights

$$E_{ii}|\psi\rangle=\nu_i|\psi\rangle\,, \qquad i=1,2,\ldots,M+N, \tag{2.32}$$

$$E_{aa}|\psi\rangle=\lambda_a|\psi\rangle\,, \qquad a=1,2,\ldots,K, \tag{2.33}$$

which for $\mathfrak{psu}(2,2|4)$ gives the two sets $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ and $\{\nu_1,\nu_2,\nu_3,\nu_4\}$. However, only six of these are independent such that the representations are characterized by the differences $\lambda_a-\lambda_{a+1}$ and $\nu_i-\nu_{i+1}$.

To specify to the real form of $\mathfrak{psu}(N,M|K)$ we require both the conjugation property and the vanishing central charge[4]

$$E_{mn}^{\dagger}\stackrel{!}{=}(-1)^{c_m+c_n}E_{nm}\,, \qquad\qquad C=\sum_m E_{mm}\stackrel{!}{=}0\,. \tag{2.34}$$

A representation of such a Lie algebra consists of a vector space $\mathcal{V}$ for which there exists a map $E_{mn}\to\pi(E_{mn})\in\mathrm{End}(\mathcal{V})$ that respects the commutation relations (2.31). As usual we refer to both the module $\mathcal{V}$ and the map π as the representation and to irreducible representations, i.e. such that have no non-trivial invariant subspaces, as multiplets.

2.3.1 Highest weight states and gradings

We will be concerned with highest weight representations which are defined in terms of a highest weight state (HWS) $|\Psi\rangle$, satisfying

$$E_{mn}|\Psi\rangle=0\,, \qquad \forall n>m\,. \tag{2.35}$$

[4] We sometimes use the notation $\stackrel{!}{=}$ for equalities or conditions to emphasize it is something we impose.

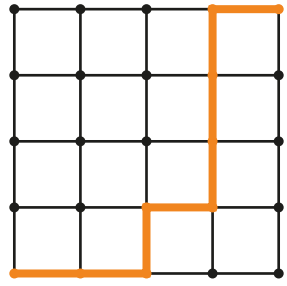


Figure 2.2. The grading can be illustrated as a path on a lattice where the nth line is vertical if $p_n = 1$ and horizontal if $p_n = 0$. Naturally, this corresponds to the locations of the hatted indices such that this grading is $12\hat{1}3\hat{2}\hat{3}\hat{4}4$ or **2333** in the notation of paper IV.

Such a definition naturally depends on how we order the labeling indices. We can specify this by writing out the p-grading of the indices such that those with $p_m = 1$ are indicated with a hat, as in $12\hat{1}3\hat{2}\hat{3}\hat{4}4$. A shorthand notation is to only indicate the positions of the hats such that the same grading would read **2333**. This can be visualized by picturing the grading as a path on a lattice illustrated in 2.2.

Starting from a HWS, the rest of the multiplet is obtained through the action of the generators. It is hence characterized by the grading and the weights of the HWS.

Combining the HWS property with required unitarity gives some bounds on the allowed weights. They can be straight-forwardly proven by starting from $\langle\Psi|[E_{mn}, E_{nm}\}|\Psi\rangle$ and using equations (2.31), (2.34) and (2.35) together with the positivity of the norm. The bounds for $\mathfrak{psu}(2,2|4)$ are

$$\lambda_a - \lambda_{a+1} \geq 0\,, \tag{2.36a}$$

$$\nu_4 \geq \nu_3 \geq \nu_1 \geq \nu_2\,, \tag{2.36b}$$

$$\begin{cases} \lambda_a + \nu_k \leq 0\,, & k = 1,2, \\ \lambda_a + \nu_k \geq 0\,, & k = 3,4. \end{cases} \tag{2.36c}$$

In the grading **2222**, we have the lowering, Cartan and raising operators

$$\begin{aligned} \text{lowering}: &\quad \left\{P_{\alpha\dot{\alpha}},\, Q^a_\alpha,\, \dot{Q}_{a\dot{\alpha}},\, L^\alpha{}_\beta\ (\alpha \geq \beta),\, \dot{L}^{\dot{\alpha}}{}_{\dot{\beta}}\ (\dot{\alpha} \geq \dot{\beta}),\, R^a{}_b\ (a \geq b)\right\} \\ \text{Cartan}: &\quad \left\{D,\, L^\alpha{}_\beta\ (\alpha = \beta),\, \dot{L}^{\dot{\alpha}}{}_{\dot{\beta}}\ (\dot{\alpha} = \dot{\beta}),\, R^a{}_b\ (a = b)\right\} \\ \text{raising}: &\quad \left\{K_{\alpha\dot{\alpha}},\, S^\alpha_a,\, \dot{S}^{a\dot{\alpha}},\, L^\alpha{}_\beta\ (\alpha \leq \beta),\, \dot{L}^{\dot{\alpha}}{}_{\dot{\beta}}\ (\dot{\alpha} \geq \dot{\beta}),\, R^a{}_b\ (a \leq b)\right\}. \end{aligned} \tag{2.37}$$

2.3.2 The oscillator formalism

Any $GL(N,M|K)$ representation with integer weights can be formulated in what is called Jordan-Schwinger oscillators. Specifying that formalism for

$\mathfrak{psu}(2,2|4)$, we have

$$E_{mn} = \bar{\chi}_m \chi_n \tag{2.38}$$

with

$$E_{\dot{\alpha}\dot{\beta}} = -\mathbf{b}_{\dot{\alpha}}\mathbf{b}^\dagger_{\dot{\beta}}, \qquad [\mathbf{b}_{\dot{\alpha}}, \mathbf{b}^\dagger_{\dot{\beta}}] = \delta_{\dot{\alpha}\dot{\beta}}, \tag{2.39}$$

$$E_{(2+\alpha)(2+\beta)} = \mathbf{a}_\alpha \mathbf{a}^\dagger_\beta, \qquad [\mathbf{a}_\alpha, \mathbf{a}^\dagger_\beta] = \delta_{\alpha\beta}, \tag{2.40}$$

$$E_{(4+a)(4+b)} = \mathbf{f}^\dagger_a \mathbf{f}_b, \qquad \{\mathbf{f}_a, \mathbf{f}^\dagger_b\} = \delta_{ab}, \tag{2.41}$$

where $\dot{\alpha}, \dot{\beta}, \alpha, \beta = 1,2$ and $a, b = 1,2,3,4$. We see that $\mathbf{a}$ and $\mathbf{b}$ are bosonic while $\mathbf{f}$ is fermionic and that all of them satisfy the corresponding oscillator algebra. All other commutators are zero. The full matrix of operators is

$$E_{mn} = \left(\begin{array}{cccc|cccc}
-\mathbf{b}_1\mathbf{b}^\dagger_1 & -\mathbf{b}_1\mathbf{b}^\dagger_2 & -\mathbf{b}_1\mathbf{a}_1 & -\mathbf{b}_1\mathbf{a}_2 & -\mathbf{b}_1\mathbf{f}_1 & -\mathbf{b}_1\mathbf{f}_2 & -\mathbf{b}_1\mathbf{f}_3 & -\mathbf{b}_1\mathbf{f}_4 \\
-\mathbf{b}_2\mathbf{b}^\dagger_1 & -\mathbf{b}_2\mathbf{b}^\dagger_2 & -\mathbf{b}_2\mathbf{a}_1 & -\mathbf{b}_2\mathbf{a}_2 & -\mathbf{b}_2\mathbf{f}_1 & -\mathbf{b}_2\mathbf{f}_2 & -\mathbf{b}_2\mathbf{f}_3 & -\mathbf{b}_2\mathbf{f}_4 \\
\mathbf{a}^\dagger_1\mathbf{b}^\dagger_1 & \mathbf{a}^\dagger_1\mathbf{b}^\dagger_2 & \mathbf{a}^\dagger_1\mathbf{a}_1 & \mathbf{a}^\dagger_1\mathbf{a}_2 & \mathbf{a}^\dagger_1\mathbf{f}_1 & \mathbf{a}^\dagger_1\mathbf{f}_2 & \mathbf{a}^\dagger_1\mathbf{f}_3 & \mathbf{a}^\dagger_1\mathbf{f}_4 \\
\mathbf{a}^\dagger_2\mathbf{b}^\dagger_1 & \mathbf{a}^\dagger_2\mathbf{b}^\dagger_2 & \mathbf{a}^\dagger_2\mathbf{a}_1 & \mathbf{a}^\dagger_2\mathbf{a}_2 & \mathbf{a}^\dagger_2\mathbf{f}_1 & \mathbf{a}^\dagger_2\mathbf{f}_2 & \mathbf{a}^\dagger_2\mathbf{f}_3 & \mathbf{a}^\dagger_2\mathbf{f}_4 \\
\hline
\mathbf{f}^\dagger_1\mathbf{b}^\dagger_1 & \mathbf{f}^\dagger_1\mathbf{b}^\dagger_2 & \mathbf{f}^\dagger_1\mathbf{a}_1 & \mathbf{f}^\dagger_1\mathbf{a}_2 & \mathbf{f}^\dagger_1\mathbf{f}_1 & \mathbf{f}^\dagger_1\mathbf{f}_2 & \mathbf{f}^\dagger_1\mathbf{f}_3 & \mathbf{f}^\dagger_1\mathbf{f}_4 \\
\mathbf{f}^\dagger_2\mathbf{b}^\dagger_1 & \mathbf{f}^\dagger_2\mathbf{b}^\dagger_2 & \mathbf{f}^\dagger_2\mathbf{a}_1 & \mathbf{f}^\dagger_2\mathbf{a}_2 & \mathbf{f}^\dagger_2\mathbf{f}_1 & \mathbf{f}^\dagger_2\mathbf{f}_2 & \mathbf{f}^\dagger_2\mathbf{f}_3 & \mathbf{f}^\dagger_2\mathbf{f}_4 \\
\mathbf{f}^\dagger_3\mathbf{b}^\dagger_1 & \mathbf{f}^\dagger_3\mathbf{b}^\dagger_2 & \mathbf{f}^\dagger_3\mathbf{a}_1 & \mathbf{f}^\dagger_3\mathbf{a}_2 & \mathbf{f}^\dagger_3\mathbf{f}_1 & \mathbf{f}^\dagger_3\mathbf{f}_2 & \mathbf{f}^\dagger_3\mathbf{f}_3 & \mathbf{f}^\dagger_3\mathbf{f}_4 \\
\mathbf{f}^\dagger_4\mathbf{b}^\dagger_1 & \mathbf{f}^\dagger_4\mathbf{b}^\dagger_2 & \mathbf{f}^\dagger_4\mathbf{a}_1 & \mathbf{f}^\dagger_4\mathbf{a}_2 & \mathbf{f}^\dagger_4\mathbf{f}_1 & \mathbf{f}^\dagger_4\mathbf{f}_2 & \mathbf{f}^\dagger_4\mathbf{f}_3 & \mathbf{f}^\dagger_4\mathbf{f}_4
\end{array}\right), \tag{2.42}$$

where we have written it out for a later comparison with the β-deformed theory below.

The oscillators act on a Fock space vacuum $|0\rangle$ defined by

$$\mathbf{a}_\alpha|0\rangle = \mathbf{b}_{\dot{\alpha}}|0\rangle = \mathbf{f}_a|0\rangle = 0. \tag{2.43}$$

The generators of the $\mathfrak{psu}(2,2|4)$ algebra (2.37) are in the oscillator formulation[5]

$$L^\alpha{}_\beta = \mathbf{a}^\dagger_\beta \mathbf{a}^\alpha - \tfrac{1}{2}\delta^\alpha_\beta \mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma, \qquad Q^a_\alpha = \mathbf{a}^\dagger_\alpha \mathbf{f}^a, \qquad S^\alpha_a = \mathbf{f}^\dagger_a \mathbf{a}^\alpha, \tag{2.44}$$

$$\dot{L}^{\dot{\alpha}}{}_{\dot{\beta}} = \mathbf{b}^\dagger_{\dot{\beta}} \mathbf{b}^{\dot{\alpha}} - \tfrac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} \mathbf{b}^\dagger_{\dot{\gamma}} \mathbf{b}^{\dot{\gamma}}, \qquad \dot{Q}_{a\dot{\alpha}} = \mathbf{b}^\dagger_{\dot{\alpha}} \mathbf{f}^\dagger_a, \qquad \dot{S}^{a\dot{\alpha}} = \mathbf{f}^a \mathbf{b}^{\dot{\alpha}}, \tag{2.45}$$

$$R^a{}_b = \mathbf{f}^\dagger_b \mathbf{f}^a - \tfrac{1}{4}\delta^a_b \mathbf{f}^\dagger_c \mathbf{f}^c, \qquad P_{\alpha\dot{\alpha}} = \mathbf{a}^\dagger_\alpha \mathbf{b}^\dagger_{\dot{\alpha}}, \qquad K^{\alpha\dot{\alpha}} = \mathbf{a}^\alpha \mathbf{b}^{\dot{\alpha}}, \tag{2.46}$$

with the dilatation operator

$$D = 1 + \tfrac{1}{2}\mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \tfrac{1}{2}\mathbf{b}^\dagger_{\dot{\gamma}} \mathbf{b}^{\dot{\gamma}}. \tag{2.47}$$

[5]The change of the index locations here is to bridge the different notations used previously in the chapter and to the one used in Paper IV.

Table 2.1. *The singleton representation of $\mathcal{N}=4$ SYM fields in the oscillator formalism.*

scalars	fermions	field strength	covariant derivative
$\Phi_{ab} \widehat{=} \mathbf{f}_a^\dagger \mathbf{f}_b^\dagger \lvert 0\rangle$	$\psi_{a\alpha} \widehat{=} \mathbf{f}_a^\dagger \mathbf{a}_\alpha^\dagger \lvert 0\rangle$ $\bar{\psi}_{a\dot{\alpha}} \widehat{=} \epsilon_{abcd} \mathbf{f}_b^\dagger \mathbf{f}_c^\dagger \mathbf{f}_d^\dagger \mathbf{b}_{\dot{\alpha}}^\dagger \lvert 0\rangle$	$F_{\alpha\beta} \widehat{=} \mathbf{a}_\alpha^\dagger \mathbf{a}_\beta^\dagger \lvert 0\rangle$ $\bar{F}_{\dot{\alpha}\dot{\beta}} \widehat{=} \mathbf{b}_{\dot{\alpha}}^\dagger \mathbf{b}_{\dot{\beta}}^\dagger \mathbf{f}_1^\dagger \mathbf{f}_2^\dagger \mathbf{f}_3^\dagger \mathbf{f}_4^\dagger \lvert 0\rangle$	$D_{\alpha\dot{\alpha}} \widehat{=} \mathbf{a}_\alpha^\dagger \mathbf{b}_{\dot{\alpha}}^\dagger$

2.3.3 The representations of $\mathcal{N}=4$ SYM fields and operators

The observables in $\mathcal{N}=4$ SYM are gauge invariant operators constructed from gauge covariant field combinations over which the trace is taken. The gauge field A_μ can thus only enter through the field strength or the covariant derivative

$$F_{\alpha\beta\dot{\alpha}\dot{\beta}} = F_{\mu\nu}(\sigma^\mu)_{\alpha\beta}(\sigma^\nu)_{\dot{\alpha}\dot{\beta}} = -\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta} - \sqrt{2}\epsilon_{\alpha\beta} F_{\dot{\alpha}\dot{\beta}}\,, \tag{2.48}$$

$$D_{\alpha\dot{\alpha}} = D_\mu(\sigma^\mu)_{\alpha\dot{\alpha}}\,, \tag{2.49}$$

where we have also defined the self-dual $F_{\alpha\beta}$ and anti-self-dual field strengths $\bar{F}_{\dot{\alpha}\dot{\beta}}$, using the standard anti-symmetric two-dimensional symbols $\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = \epsilon^{21} = \epsilon^{\dot{2}\dot{1}} = 1$. Due to the Bianchi identity $D_{[\mu}F_{\mu\nu]} = 0$ and the equations of motion, it is possible to write any gauge covariant combination of the $\mathcal{N}=4$ SYM field content symmetrized in all spinor indices in terms of the building blocks in table 2.3.3.

This constitutes the singleton representation of $\mathfrak{psu}(2,2|4)$. Notice how the covariant derivative can be applied to any of the Fock states an infinite number of times, reflecting the non-compactness of $\mathfrak{psu}(2,2|4)$.

Composite operators are built by taking traces of field combinations in the singleton representation. This corresponds to tensor products in the oscillator formalism, e.g.

$$\mathrm{tr}(ZZX) \;\widehat{=}\; \mathbf{f}_1^\dagger \mathbf{f}_2^\dagger \lvert 0\rangle \otimes \mathbf{f}_1^\dagger \mathbf{f}_2^\dagger \lvert 0\rangle \otimes \mathbf{f}_1^\dagger \mathbf{f}_3^\dagger \lvert 0\rangle\,. \tag{2.50}$$

The symmetry generators act as usual on these tensor products according to the rule $A(a\otimes b) = (Aa)\otimes b + a\otimes(Ab)$.

Given a grading, we can label the operators by their oscillator content using the number operators, e.g. $n_{\mathbf{a}_\alpha} = \mathbf{a}_\alpha^\dagger \mathbf{a}_\alpha$. We will use the notation

$$\mathbf{n} = [n_{\mathbf{b}_1}, n_{\mathbf{b}_2} | n_{\mathbf{f}_1}, n_{\mathbf{f}_2}, n_{\mathbf{f}_3}, n_{\mathbf{f}_4} | n_{\mathbf{a}_1}, n_{\mathbf{a}_2}]\,. \tag{2.51}$$

In this notation, the central charge acting on a single site is

$$C = -2 - n_{\mathbf{b}} + n_{\mathbf{f}} + n_{\mathbf{a}}\,, \tag{2.52}$$

with $n_{\mathbf{a}} = n_{\mathbf{a}_1} + n_{\mathbf{a}_2}$, etc., and its vanishing is a requirement on each state for it to be part of the singleton representation.

2.3.4 The conformal field theory language

An operator $\mathcal{O}$ in a (supersymmetric) CFT that maps to a HWS of the symmetry group is called a (super)conformal primary, i.e. such operators that are annihilated by all the raising[6] generators in (2.37). Acting on the primary operator with the lowering generators builds up the entire representation and all these operators are called (conformal) descendants. It can be shown that all correlators in the CFT can be expressed in terms of the primary operators and they are hence the central objects of study.

A very important subset of these is the chiral primary operators, or BPS-operators. Their defining property is that they are annihilated by one or more of the raising operators Q^a_α,

$$[Q^a_\alpha, \mathcal{O}\} = 0\,. \tag{2.53}$$

As a consequence, there is a relation between the quantum numbers of the operator. In particular, its conformal dimension Δ is expressible through the spin and R-symmetry quantum numbers and is hence required to be a half-integer. As such, Δ cannot receive quantum corrections and the operator is said to be protected.

2.4 Correlation functions

Correlation functions in a CFT are very restricted due to the large symmetry. One-point functions are required to be constant, normally zero, and the two-point functions need to be on the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{(x_1-x_2)^{\Delta_1+\Delta_2}}\,, \tag{2.54}$$

where the denominator is a shorthand notation for $\left((x_1-x_2)^2\right)^{\frac{\Delta_1+\Delta_2}{2}}$. The normalization of the operators are normally chosen to maintain this canonical form of the two-point function.

Three-point functions also have their spacetime dependence dictated by the conformal symmetry; it is

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{(x_{12})^{\Delta_1+\Delta_2-\Delta_3}(x_{23})^{\Delta_2+\Delta_3-\Delta_1}(x_{31})^{\Delta_3+\Delta_1-\Delta_2}}\,, \tag{2.55}$$

with $x_{ij} = x_i - x_j$ and where λ_{123} are the field dependent structure constants coming from the operator product expansion. With the normalization of the two-point functions (2.54), these are well-defined theory dependent numbers.

[6]The nomenclature is a bit confusing as the lowering operators *increase* the dimension Δ and what is referred to as highest/lowest state differ between references. We choose to name the symmetry generators according to (2.37) and to call the primary operators HWS.

As mentioned in the introduction, these, together with the set of conformal dimensions and corresponding primary operators, constitute the conformal data of the theory.

First at the level of four-point functions may the correlators depend non-trivially on the spacetime coordinate but only through the conformally invariant cross-ratios

$$\frac{(x_1-x_2)^2(x_3-x_4)^2}{(x_1-x_3)^2(x_2-x_4)^2} \qquad \text{and} \qquad \frac{(x_1-x_4)^2(x_2-x_3)^2}{(x_1-x_3)^2(x_2-x_4)^2}\,. \tag{2.56}$$

2.5 Renormalization and anomalous dimension

At quantum-level, the correlation functions of section 2.4 become infinite and renormalization with some regularization scheme is required to render them finite. In $\mathcal{N}=4$ SYM, all such infinites can be mended through

$$O_I^{\text{ren}} = \mathcal{Z}_{IJ} O_J^{\text{bare}}\,, \tag{2.57}$$

where I, J are some collective indices that label the operators and $\mathcal{Z}$ is the matrix controlling the field renormalization. $\mathcal{Z}$ depends on the coupling g and the regularization parameter Λ and, as it is a matrix, we see from (2.57) that the renormalized operators are linear combinations of the bare operators, a phenomenon called operator mixing.

The conformal dimensions develop a non-half-integer part in this process,

$$\Delta = \Delta_0 + \gamma\,, \tag{2.58}$$

where $\gamma = \sum_n g^{2n}\gamma^{(n)}$ is the anomalous dimension expanded[7] in the coupling. The two-point function for an operator with itself then becomes

$$\langle O^{\text{ren}}(x_1)\bar{O}^{\text{ren}}(x_2)\rangle = \frac{1}{(x_1-x_2)^{2\Delta_0}} - g^2\gamma^{(1)}\log|\Lambda(x_1-x_2)| + \dots\,. \tag{2.59}$$

The dilatation operator is hence now a matrix, related to $\mathcal{Z}$ through

$$D = D^{(0)} + D_\gamma = D^{(0)} + \mathcal{Z}^{-1}\frac{\mathrm{d}}{\mathrm{d}\log\Lambda}\mathcal{Z}\,. \tag{2.60}$$

Consequently, only operators in a basis that diagonalizes D_γ have a well-defined dimension γ. The task to find them and their corresponding dimensions is the spectral problem of $\mathcal{N}=4$ SYM.

[7] In general there may also be odd powers of g in this expansion but the even powers cover all cases considered in this thesis.

2.5.1 The spin-chain picture

The dilatation operator can be expanded in the coupling,

$$D = \sum_{n=0}^{\infty} g^{2n} D^{(n)}. \tag{2.61}$$

At the nth loop order, a maximum of $n+1$ fields inside a local operator can be involved in a connected interaction and, in the planar limit, they need to be neighbors inside the trace in order to not be suppressed. This means that at each order, the dilatation operator acts on an operator $\mathcal{O}_L$ as a sum over local densities

$$D^{(n)} = \sum_{l=1}^{L} D^{(n)}_{l,\dots,l+n}, \tag{2.62}$$

where the cyclic trace in $\mathcal{O}_L$ identifies position $l+L$ with l.

This form is very suggestive of a spin-chain interpretation, especially with the tensor product representation (2.50) in mind. This was indeed the realization when the action of $D^{(1)}$ on single-trace operators of scalars was diagramatically computed in [1]. There it was first observed that $D^{(1)}_{\text{scal}}$ acts as the Hamiltonian of an integrable spin-chain, specifically the XXX spin-chain with $SO(6)$-symmetry. Explicitly,

$$D^{(1)}_{\text{scal.}} = 2\sum_{l=1}^{L} H_{l,l+1}, \qquad H_{l,l+1} = \mathbb{1} - \mathbb{P}_{l,l+1} + \frac{1}{2}\mathbb{K}_{l,l+1}, \tag{2.63}$$

where $\mathbb{P}$ and $\mathbb{K}$ are the permutation and trace operators, respectively,

$$\mathbb{P}(a\otimes b) = b\otimes a, \tag{2.64}$$

$$\mathbb{K}(a\otimes b) = a\cdot b\sum_{i=1}^{6} \hat{e}_i\otimes\hat{e}_i, \tag{2.65}$$

with $\hat{e}_i$ being the basis vectors in $\mathbb{R}^6$.

This result was later generalized in [24] to the full 1-loop dilatation operator $D^{(1)}$ and consecutively given a $PSU(2,2|4)$-spin-chain interpretation in [25]. Its action on an operator of length L can elegantly be written in terms of projectors $\Pi^{(j)}$ onto the irreducible length-2 representations $\mathcal{V}_j$ of $PSU(2,2|4)$ in the decomposition of the tensor product of two singleton representations

$$\mathcal{V}\otimes\mathcal{V} = \bigoplus_j \mathcal{V}_j. \tag{2.66}$$

The 1-loop dilation operator then reads

$$D^{(1)} = 2\sum_{l=1}^{L}\sum_{j=0}^{\infty} h(j)\Pi^{(j)}_{l,l+1}, \tag{2.67}$$

where $h(j)$ are the harmonic numbers

$$h(j) = \sum_{k=1}^{j} \frac{1}{k} . \tag{2.68}$$

The extension of the spin-chain picture to higher loop orders inevitably reaches a limit at the Lth loop. At that point, the interaction length is longer than the length of the chain and the sum over local densities is not a valid expression for the dilatation operator. This finite-size effect is called wrapping and requires special care in the perturbative calculations of anomalous dimensions. Furthermore, beyond 1-loop interactions may change the number of fields and hence the spin-chain length; such chains are called dynamical. [24–30]

2.5.2 Subsectors

D commutes with $L^{\alpha}{}_{\beta}$ and $R^{a}{}_{b}$, as seen in section 2.2, which implies that the operator mixing can only occur between operators with the same quantum numbers, thus creating closed subsectors. The most relevant for this thesis is the $SU(2)$-subsector which consists of operators built out of two $SU(4)$-scalars. For definiteness we will choose Z and X such that the quantum numbers for an operator with $L-M$ scalars of type Z and M of type X has the quantum numbers $(L,0,0;L-M,M,0)$. In the oscillator formalism it is labeled by $\mathbf{n} = [0,0|L,L-M,M,0|0,0]$ with $\mathrm{tr}(ZZ\dots Z)$ being the HWS. Note that $\mathrm{tr}(ZZ\dots Z)$ is a chiral primary and has vanishing anomalous dimension. The 1-loop dilatation operator acts on this subsector as

$$D^{(1)}_{SU(2)} = 2\sum_{l=1}^{L} (\mathbb{1} - \mathbb{P}_{l,l+1}) , \tag{2.69}$$

which is the Hamiltonian of the Heisenberg $XXX_{1/2}$ spin-chain. We will study it carefully in chapter 5.

Another notable subsector is $SU(3|2)$, consisting of the three scalars Z, X and Y with the combined quantum numbers $(3,0,0;1,1,1)$ and two fermions with quantum numbers $(\frac{3}{2},\pm\frac{1}{2},0;\frac{1}{2},\frac{1}{2},\frac{1}{2})$. We see that these quantum numbers allow for the mentioned length changing mixing such that L is no longer a useful label for the operators. This can, however, only occur at two or higher loops so L is still meaningful in one-loop calculations.

Similarly and accordingly, the one-loop $SO(6) \simeq SU(4)$-subsector of scalars in [1] is no longer closed at higher loops and mixes with the entire $PSU(2,2|4)$-representation.

The last subsector that we highlight is the $SU(1,1)$-subsector built from a number, possibly infinite, of covariant derivatives D_{12} distributed over L scalars Z. This sector is closed at all loops and is also referred to as the $SL(2)$-subsector.

2.5.3 The Konishi multiplet

One of the most well-studied multiplets in $\mathcal{N} = 4$ SYM is the Konishi multiplet. Its operator with the lowest dimension is

$$\mathrm{tr}(Z\bar{Z} + X\bar{X} + Y\bar{Y}), \tag{2.70}$$

which is the simplest operator with a non-vanishing anomalous dimension. The latter is currently known up to an impressive eleven loops, as reported in [5], using perturbative algorithm of the Quantum Spectral Curve. We will return to that in chapter 7. The operator (2.70) is the HWS in the grading **2222** and acting on it with the symmetry operators generates the full multiplet.

However, there is a subtlety as the full Konishi multiplet consists of operators of different lengths and, as mentioned in the previous section, these operators are only connected by the full symmetry generators at finite coupling. For $g = 0$, the Konishi multiplets splits up into four multiplets, one with $L = 2$, two with $L = 3$ and one with $L = 2$, in an effect known as shortening.

It was the goal of Paper IV to study the splitting of this multiplet in the β-deformation, to be introduced below in chapter 3, and to calculate some of its anomalous dimensions.

The two-point function of the Konishi operator (2.70) and any operator in the $SU(2)$-subsector was calculated in Paper I inside a version of $\mathcal{N} = 4$ SYM with a codimensional-one defect, which we will introduce in chapter 6.

2.6 The AdS/CFT correspondence

The global symmetries $SO(2,4) \times SO(6)$ of $\mathcal{N} = 4$ SYM and speculative ideas about a holographic principle lead Maldacena to a remarkable conjecture in 1997 called the AdS/CFT correspondence [31]. It states that four-dimensional $\mathcal{N} = 4$ SYM is dual to type IIB super string theory on the background geometry $AdS_5 \times S^5$. This provoked a flurry of activity and opened up a field of research that has kept pushing these ideas for more than two decades[8]. Although it is still a conjecture and no proof is within sight, it has passed an extraordinary number of tests and is now generally taken as a truth. Generalized versions of this first conjecture go collectively under the name gauge/gravity dualities. The subject is huge and reaches all the way to attempted applications in condensed matter theory. The original and archetypical example of the AdS/CFT correspondence is however our focus, with an additional deformation later on in chapter 3.

[8] For the immediate elaboration, see e.g. [32, 33].

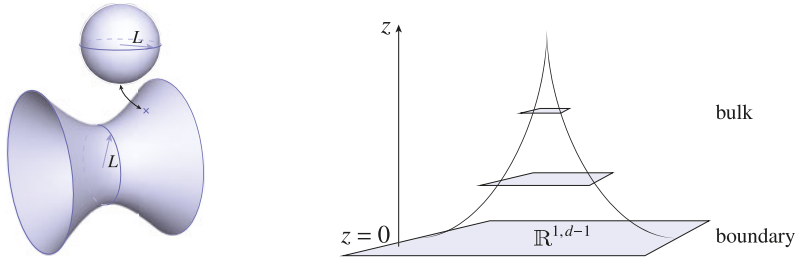


Figure 2.3. An illustration of $AdS_2 \times S^2$ to the left, both with radius L, and to right the flat space slicing of the Poincaré patch (2.72).

2.6.1 The anti-de Sitter geometry

The anti-de Sitter geometry AdS_d of d dimensions is a maximally symmetric solution to the vacuum Einstein equations with a constant negative curvature L. It can be defined through the constraint

$$-X_0^2 - X_d^2 + \sum_{i=1}^{d-1} X_i^2 = -L^2 \tag{2.71}$$

from which it is clear that it has the isometry group $SO(2,d)$. There are many common choices of coordinate systems, out of which the Poincaré patch is illustrative for our purposes. The metric in these coordinates is

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \tag{2.72}$$

where each constant $z > 0$ gives a flat space slicing. The conformal boundary at $z \to 0$ is hence the Minkowski space $\mathbb{R}^{1,d-2}$, referred to as just the boundary. $z > 0$ is called the bulk and z the radial coordinate. Two illustrations of the geometry can be found in figure 2.3.

Together with the sphere, $AdS_5 \times S^5$ has the same isometry group as the global symmetry group of $\mathcal{N} = 4$ SYM and its conformal boundary is precisely the 4D flat spacetime. The mental leap of the holographic principle of the AdS/CFT correspondence states that the physics described inside the bulk of $AdS_5 \times S^5$ is captured entirely by the 4D conformal theory, viewed as living on the boundary.

2.6.2 The correspondence

The AdS/CFT correspondence more specifically states that correlation functions of local composite operators in $\mathcal{N} = 4$ SYM can be calculated by so called boundary to boundary correlators in the bulk theory. This requires a map between the free parameters λ and N of $\mathcal{N} = 4$ SYM to counterparts in string theory.

Table 2.2. *A summary of the parameters and regimes of the AdS/CFT correspondence.*

$\mathcal{N}=4$ SYM	IIB strings on $AdS_5 \times S^5$
$\lambda = g_{\rm YM}^2 N$	$T = \frac{\sqrt{\lambda}}{2\pi}$
$N, g_{\rm YM}$	$g_{\rm s} = \frac{g_{\rm YM}^2}{4\pi} = \frac{\lambda}{4\pi N}$
planar: $N \to \infty$	$g_{\rm s} \to 0$: free strings
strong coupl.: $\lambda \gg 1$	$T \gg 1$: supergravity

The string theory is characterized by its string tension T, governing the amount of accessible sting excitations, and the string coupling constant $g_{\rm s}$ which determines the interaction probability of strings splitting and joining. The latter is connected to the Yang-Mills coupling through

$$g_{\rm s} = \frac{g_{\rm YM}^2}{4\pi} = \frac{\lambda}{4\pi N}\,. \tag{2.73}$$

This means that the large N limit of $\mathcal{N}=4$ SYM is mirrored by a vanishing string coupling, i.e. the planar limit corresponds to non-interacting strings.

The string tension, on the other hand, is connected to the 't Hooft coupling,

$$T = \sqrt{\lambda}/2\,. \tag{2.74}$$

At strong coupling $\lambda \gg 1$, the large string tension renders any excitation very energetic and hence prohibited. What remains is the supergravity limit of string theory. This is one of the most important features of the correspondence; it maps a strongly coupled quantum regime in one theory to a classical and weakly coupled regime in the other. It is a great virtue in that it allows us to use perturbation theory on one side to access results way out of reach from such computational methods on the other. Table 2.2 summarizes the map of the correspondence.

The strong/weak coupling duality becomes a complication though in attempting to verify the correspondence; it is difficult to obtain results on both the sides simultaneously. Various techniques have been developed to probe the strong coupling regimes and all to a perfect agreement with the conjecture. We will look closer into one of them in chapter 4.

3. Deformations of $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM has, as we have seen, an extraordinary amount of symmetry and makes a perfect testing ground for developing computational techniques. In particular its clear holographic interpretation and the integrability in the planar limit have been great successes the last two decades.

In an attempt to capitalize on these advancements, there has been a lot of work to generalize the applicability through various deformations. By introducing different types of parameters, one can alter the theory in various precise ways to capture more varied physics while retaining as much as possible of the salient features. We will introduce two types of such deformations in this chapter: the β-deformation and the $\mathcal{N} = 2^*$ SYM theory.

3.1 β-deformed $\mathcal{N} = 4$ SYM

The β-deformation was first considered among the marginal deformations in [34] and is in fact a special case of a more general deformation called the γ-deformation. A common way to introduce the full γ-deformation is to start from the $\mathcal{N} = 4$ SYM action (2.11) formulated in $\mathcal{N} = 1$ superspace and then replace all field multiplications with a non-commutative $*$-product. It is defined through the $SU(4)$ R-charges of the fields, here gathered in vectors $\mathbf{q}$, according to

$$A * B = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathbf{q}_A \wedge \mathbf{q}_B} AB, \tag{3.1}$$

where $\wedge$ carries the anti-symmetric matrix $\mathbf{C}$ of three deformation parameters

$$\mathbf{q}_A \wedge \mathbf{q}_B = \mathbf{q}_A^T \mathbf{C} \mathbf{q}_B, \qquad \mathbf{C} = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}. \tag{3.2}$$

The R-charges for the (complex) scalars and fermions are listed in table 3.1. There we have also included the basis

$$Q_1 = q_1 - q_2\,, \qquad Q_2 = q_2 - q_3\,, \qquad r = \tfrac{2}{3}(q_1 + q_2 + q_3)\,. \tag{3.3}$$

The β-deformation is the special case when all deformation parameters are equal and real,

$$\gamma_i = -\beta \in \mathbb{R}\,. \tag{3.4}$$

Table 3.1. *The non-zero $SU(4)$ R-charges for the field content in $\mathcal{N}=4$ SYM. The pluses and minuses for the fermions are all $\pm\frac{1}{2}$ but with the fraction omitted to reduce clutter. Note that ψ_4 only has non-zero r in the basis* (3.3) *and is untouched by the β-deformation.*

	ϕ^1	ϕ^2	ϕ^3	$\psi_{1\alpha}$	$\psi_{2\alpha}$	$\psi_{3\alpha}$	$\psi_{4\alpha}$
q^1	1	0	0	+	−	−	+
q^2	0	1	0	−	+	−	+
q^3	0	0	1	−	−	+	+
Q_1	1	−1	0	1	−1	0	0
Q_2	0	1	−1	0	1	−1	0
$3r$	2	2	2	−1	−1	−1	−1

$\frac{1}{2}$

The $\wedge$-product then simplifies to

$$\mathbf{q}_A \wedge \mathbf{q}_B = \beta \sum_{a,b,c=1}^{3} \varepsilon_{abc} q_A^a q_B^b = \beta \sum_{a,b=1}^{3} \varepsilon_{ab3} Q_A^a Q_B^b, \tag{3.5}$$

i.e. it is independent of the third charge r. From table 3.1 we can than see that the β-deformation leaves field products with ψ_4 untouched.

Although non-commutative, the $*$-product is associative and a multiple product evaluates to

$$A_1 * A_2 * \ldots * A_L = \exp\left[\tfrac{\mathrm{i}}{2} \sum_{i<j} \mathbf{q}_{A_i} \wedge \mathbf{q}_{A_j}\right] A_1 A_2 \ldots A_L. \tag{3.6}$$

3.1.1 The action

Substituting the $*$-product into the $\mathcal{N}=1$ action (2.11) affects only the commutators which are changed to $*$-commutators. The expansion into component fields then yields the single-trace action

$$\begin{aligned} S_{\text{s-t}} = \frac{2}{g_{\text{YM}}^2} \int \mathrm{d}^4x \ \mathrm{tr}\Big[& -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} D^\mu \bar{\phi}_i \, D_\mu \phi^i + \mathrm{i} \bar{\psi}_{\dot{\alpha}}^a (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} D_\mu \psi_{a\beta} \\ & - \frac{1}{2} \Big(\mathrm{i} \varepsilon^{ijk4} \psi_i^\alpha [\phi^j, \psi_{k\alpha}]_* + 2\mathrm{i} \psi_i^\alpha [\bar{\phi}_i, \psi_{4\alpha}] + \text{h.c.} \Big) \\ & + \frac{1}{2} \left([\phi^i, \phi^j]_* [\bar{\phi}_i, \bar{\phi}_j]_* - \frac{1}{2} [\phi^i, \bar{\phi}_i][\phi^j, \bar{\phi}_j] \right) \Big]. \end{aligned} \tag{3.7}$$

Note that the commutator involving the gaugino ψ_4 is not starred, in accordance with the comment above.

However, this procedure also creates double-trace terms due to the fact that the trace of a starred commutator is no longer zero. For gauge group $SU(N)$,

this double-trace term has the form

$$S_{\text{d-t}} = -\frac{1}{N}\int \mathrm{d}^4x\ \mathrm{tr}\Big([\phi^i,\phi^j]_*\Big)\mathrm{tr}\Big([\bar{\phi}_i,\bar{\phi}_j]_*\Big)\,, \tag{3.8}$$

but will not play any explicit role in this thesis. For discussions of the emergence and effects of new interaction terms due to the deformations, see for instance [35–37].

3.1.2 Symmetries

The β-deformation obviously breaks the R-symmetry out of which only the Cartan subalgebra $\mathfrak{u}(1)^{\times 3}$ remains. There is, however, still a discrete permutation symmetry S_3 among the three complex scalars.

Most of the supersymmetry is also broken but since both A_μ and the gaugino remain unaffected with $Q_1 = Q_2 = 0$, an $\mathcal{N} = 1$ is still realized with the 4 supercharges $Q^4_\alpha, \dot{Q}_{4\dot{\alpha}}, S^\alpha_4$ and $\dot{S}^{4\dot{\alpha}}$.

In the oscillator formalism, the broken and retained symmetry generators organize themselves according to

$$E_{mn} = \left(\begin{array}{cccc|cccc}
-\mathbf{b}_1\mathbf{b}_1^\dagger & -\mathbf{b}_1\mathbf{b}_2^\dagger & -\mathbf{b}_1\mathbf{a}_1 & -\mathbf{b}_1\mathbf{a}_2 & -\mathbf{b}_1\mathbf{f}_1 & -\mathbf{b}_1\mathbf{f}_2 & -\mathbf{b}_1\mathbf{f}_3 & -\mathbf{b}_1\mathbf{f}_4 \\
-\mathbf{b}_2\mathbf{b}_1^\dagger & -\mathbf{b}_2\mathbf{b}_2^\dagger & -\mathbf{b}_2\mathbf{a}_1 & -\mathbf{b}_2\mathbf{a}_2 & -\mathbf{b}_2\mathbf{f}_1 & -\mathbf{b}_2\mathbf{f}_2 & -\mathbf{b}_2\mathbf{f}_3 & -\mathbf{b}_2\mathbf{f}_4 \\
\mathbf{a}_1^\dagger\mathbf{b}_1^\dagger & \mathbf{a}_1^\dagger\mathbf{b}_2^\dagger & \mathbf{a}_1^\dagger\mathbf{a}_1 & \mathbf{a}_1^\dagger\mathbf{a}_2 & \mathbf{a}_1^\dagger\mathbf{f}_1 & \mathbf{a}_1^\dagger\mathbf{f}_2 & \mathbf{a}_1^\dagger\mathbf{f}_3 & \mathbf{a}_1^\dagger\mathbf{f}_4 \\
\mathbf{a}_2^\dagger\mathbf{b}_1^\dagger & \mathbf{a}_2^\dagger\mathbf{b}_2^\dagger & \mathbf{a}_2^\dagger\mathbf{a}_1 & \mathbf{a}_2^\dagger\mathbf{a}_2 & \mathbf{a}_2^\dagger\mathbf{f}_1 & \mathbf{a}_2^\dagger\mathbf{f}_2 & \mathbf{a}_2^\dagger\mathbf{f}_3 & \mathbf{a}_2^\dagger\mathbf{f}_4 \\
\hline
\mathbf{f}_1^\dagger\mathbf{b}_1^\dagger & \mathbf{f}_1^\dagger\mathbf{b}_2^\dagger & \mathbf{f}_1^\dagger\mathbf{a}_1 & \mathbf{f}_1^\dagger\mathbf{a}_2 & \mathbf{f}_1^\dagger\mathbf{f}_1 & \mathbf{f}_1^\dagger\mathbf{f}_2 & \mathbf{f}_1^\dagger\mathbf{f}_3 & \mathbf{f}_1^\dagger\mathbf{f}_4 \\
\mathbf{f}_2^\dagger\mathbf{b}_1^\dagger & \mathbf{f}_2^\dagger\mathbf{b}_2^\dagger & \mathbf{f}_2^\dagger\mathbf{a}_1 & \mathbf{f}_2^\dagger\mathbf{a}_2 & \mathbf{f}_2^\dagger\mathbf{f}_1 & \mathbf{f}_2^\dagger\mathbf{f}_2 & \mathbf{f}_2^\dagger\mathbf{f}_3 & \mathbf{f}_2^\dagger\mathbf{f}_4 \\
\mathbf{f}_3^\dagger\mathbf{b}_1^\dagger & \mathbf{f}_3^\dagger\mathbf{b}_2^\dagger & \mathbf{f}_3^\dagger\mathbf{a}_1 & \mathbf{f}_3^\dagger\mathbf{a}_2 & \mathbf{f}_3^\dagger\mathbf{f}_1 & \mathbf{f}_3^\dagger\mathbf{f}_2 & \mathbf{f}_3^\dagger\mathbf{f}_3 & \mathbf{f}_3^\dagger\mathbf{f}_4 \\
\mathbf{f}_4^\dagger\mathbf{b}_1^\dagger & \mathbf{f}_4^\dagger\mathbf{b}_2^\dagger & \mathbf{f}_4^\dagger\mathbf{a}_1 & \mathbf{f}_4^\dagger\mathbf{a}_2 & \mathbf{f}_4^\dagger\mathbf{f}_1 & \mathbf{f}_4^\dagger\mathbf{f}_2 & \mathbf{f}_4^\dagger\mathbf{f}_3 & \mathbf{f}_4^\dagger\mathbf{f}_4
\end{array}\right), \tag{3.9}$$

where the broken symmetries are grayed out.

Although a general γ-deformation spoils the conformal invariance, it is retained in the β-deformation[1], also at the quantum level. The double-trace terms mentioned above are needed but do still have vanishing β-function.

3.1.3 Holographic dual

The β-deformation has a holographic dual first found in [38], known since as the Lunin-Maldacena background for string theory. Similarly to the gauge theory side, it is obtained by breaking the $SO(6)$-symmetry which amounts to a deformation of the five-sphere. The deformation can be achieved by sequentially performing a T-duality transformation, a shift and another T-duality transformation back again, called a TsT-transformation. The AdS-part of the geometry stays the same, reflecting the preserved conformal invariance.

[1] This is true when the gauge group is $SU(N)$ which is the case we consider here. See [35].

3.2 $\mathcal{N} = 2^*$ SYM

The second and very different deformation we consider in this thesis is $\mathcal{N} = 2^*$ SYM. It is the unique relevant deformation of $\mathcal{N} = 4$ SYM that still has $\mathcal{N} = 2$ supersymmetry. It is obtained by first rearranging the $\mathcal{N} = 4$ SYM field content into $\mathcal{N} = 2$ multiplets. The gauge field, one complex scalar and two fermions make up the vector multiplet while the remaining two complex scalars and fermions compose a complex hypermultiplet:

$$\text{vector multiplet} : \{A_\mu, Y, \psi_3, \psi_4\} \tag{3.9}$$

$$\text{massive hyper} : \{X, Z, \psi_1, \psi_2\} \tag{3.10}$$

Secondly, the matter in the hypermultiplet is given a mass M which adds regular quadratic mass terms for the fields in the hypermultiplet plus cubic couplings for the scalars.

The introduction of a mass scale naturally breaks the conformal symmetry, its presence is indicated by the star, but the $\mathcal{N} = 2$ supersymmetry is retained such that eight supercharges survive.

The R-symmetry is also obviously broken. In flat space, an $SU(2) \times U(1)$-symmetry remains but as we later will be considering this theory on S^4 it is further broken down to $U(1) \times U(1)$ [39].

The two different limits for the mass M sends $\mathcal{N} = 2^*$ either to $\mathcal{N} = 4$ SYM for vanishing mass or to pure $\mathcal{N} = 2$ SYM, when the mass is large and the heavy hypermultiplet may be integrated out. We will see consequences of this in the next chapter as we consider the results of localization applied to this theory.

3.2.1 Holographic dual

Despite $\mathcal{N} = 2^*$ not being conformal, there is a known holographic dual and it is thus an example of a more general gauge/gravity duality than the original AdS/CFT correspondence. It is called the Pilch-Warner background after the discoverers in [40] and is a deformation of both the two geometry factors AdS_5 and S^5, which is to be expected from the broken conformality and R-symmetry.

The first string correction to this background was calculated in [41] and successfully compared to the strong coupling results for $\mathcal{N} = 2^*$ as another confirmation of the duality.

Paper III drew inspiration from the brane calculation in [42] to push the knowledge about $\mathcal{N} = 2^*$ SYM further and open up for even more detailed comparisons with future calculations in the Pilch-Warner background.

4. Localization and matrix models

The word "local" has many meanings depending on the context. In supersymmetric field theories, localization refers to a rare but remarkable possibility of reducing the partition function into an integration over a small subset of the field configurations, possibly even a finite dimensional one. This is an enormous simplification that allows for exact results inaccessible by any other means.

Such a simplification was first conjectured in the calculation of the supersymmetric Wilson loop in [43] and further substantiated in [44]. It was finally proven by Pestun in [45] and has provided a lot of results since.

Let us here sketch the principle. Start with the partition function of the theory,

$$Z = \int D\phi \, \mathrm{e}^{-S[\phi]}, \tag{4.1}$$

where ϕ collectively denotes the entire set of fields, $S[\phi]$ is the Euclidean action and $D\phi$ is the integration measure for the space of all possible field configurations.

Suppose now that there is a fermionic symmetry δ, i.e. $\delta S[\phi] = 0$, which also leaves the measure invariant. Use this to perturb the action together with some (usually fermionic) field dependent quantity $V[\phi]$ such that we get the deformed partition function

$$Z_t = \int D\phi \, \mathrm{e}^{-S[\phi]-t\delta V}, \tag{4.2}$$

controlled by the deformation parameter t.

Taking the derivate with respect to this t shows, however, that this is "control" in the meekest sense; the partition function Z_t is in fact independent of t. Assuming vanishing fields at the boundary, we have

$$\partial_t Z_t = \int D\phi \, \delta V \, \mathrm{e}^{-S-t\delta V} = \int D\phi \, \delta\left(V \, \mathrm{e}^{-S-t\delta V}\right) = 0, \tag{4.3}$$

where we have used the invariance of the measure and the action and that $\delta^2 V = 0$.

This allows us to choose any value for t. If the bosonic part of δV is positive definite, we can in particular let $t \to \infty$ with the effect that the partition function is dominated by the contribution at the field configurations for which $\delta V[\phi] =$

0. These loci are often finite dimensional spaces of solutions ϕ_* and the full partition function is hence localized to a finite dimensional integral.

Including the fluctuations around ϕ_* gives quadratic terms in the exponent for which a functional generalization of Gaussian integration can be carried out. The resulting expression is called the one-loop determinant $Z_{\text{1-loop}}$, consisting of the functional determinants depending on the fields and their interactions and thus capturing the dynamics of the theory.

Such determinants are in general divergent and, in order to render the expression finite, the theory needs to be defined on a compact space where the operator spectra are discrete. The cancellation between fermions in the numerator and bosons in the denominator is then ensured by supersymmetry.

What is left in the localized partition function is

$$Z = \int \mathrm{d}\phi_* \, \mathrm{e}^{-S[\phi_*]} Z_{\text{1-loop}}[\phi_*] \,. \tag{4.4}$$

Notice that the same analysis can be made for any operator $\mathcal{O}$ with δ-invariance inserted into the partition function where a common example would be the mentioned expectation value of a supersymmetric Wilson loop.

4.1 Actions on the sphere

As mentioned just above, we need theories defined on a compact spacetime in order to use localization. We thus put $\mathcal{N} = 4$ and $\mathcal{N} = 2^*$ SYM onto the four-sphere S^4 of radius R. Still being interested mainly in the flat space theories, we will then take the decompactification limit $R \to \infty$ to regain flat space, once the localization results have been obtained.

The curved spacetime necessarily alter the actions as conformal invariance requires a coupling for the scalars to the now non-zero Ricci curvature. Generally for a sphere S^d, we have $\mathcal{R} = d(d-1)/R^2$ and the scalar coupling is

$$\mathcal{L}_{\text{Ric.}} = -\frac{2}{g_{\text{YM}}^2} \, \mathrm{tr} \, \frac{\mathcal{R}}{6} \varphi_i \varphi_i \,. \tag{4.5}$$

Naturally, the measure on the sphere also contains the determinant of the metric, $\mathrm{d}^4x \sqrt{g}$.

Moreover, to ensure the off-shell supersymmetry that underlies the localization procedure, further auxiliary fields are added to the action. Effectively, this multiplies the Ricci coupling (4.5) with a factor of $3/2$ which together with the volume $8\pi^2R^2/3$ of S^4 will make up the prefactor in the exponentials that will appear below. See [45] for a thorough derivation.

4.2 Large N matrix models

The localized partition functions for the theories of our interest are matrix models. In general, such models consist of a symmetry group, an ensemble of matrices and a probability function. They were first studied by Wigner in the context of the energy levels of nuclei but have later found many different applications.

Let us introduce the concepts through one of the simplest examples, the Gaussian Unitary Ensemble, which happens to be relevant for $\mathcal{N} = 4$ SYM.

The partition function for a random Hermitian matrix M, invariant under unitary transformations and with a quadratic potential $V \propto \operatorname{tr} M^2$, reads

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-NV(M)} = \int \mathrm{d}^N a \, \Delta(a) \mathrm{e}^{-\frac{8\pi^2 N}{\lambda} \sum_{i=1}^N a_i^2} = \int \mathrm{d}^N a \, \mathrm{e}^{-S[a]}, \tag{4.6}$$

where a_i are the eigenvalues of the matrix, $\Delta(a) = \prod_{i<j}(a_i - a_j)^2$ is the Vandermonde determinant entering as the Jacobian of the diagonalizing transformation and the integration runs over the real values of a_i. We have chosen the proportionality constant of the potential in the Boltzmann factor for reference in the next section.

We consequently have the action

$$S[a] = 2 \sum_{i<j} \log|a_i - a_j| - \frac{8\pi^2}{\lambda} N \sum_{i=1}^N a_i^2 \tag{4.7}$$

which can be interpreted as a one-dimensional static system of particles experiencing a quadratic potential and a repulsive Coulomb force, i.e. the eigenvalues behave as a 1D Coulomb gas.

We may now take the large N limit and solve the resulting saddle-point equations for $S[a]$. To this end, we introduce the density function for the eigenvalues

$$\rho(x) = \frac{1}{N} \sum_{i=1}^N \delta(x - a_i), \qquad \text{with} \qquad \int_{-\mu}^{\mu} \mathrm{d}x \, \rho(x) = 1, \tag{4.8}$$

where $\pm\mu$ are the largest and smallest eigenvalues, respectively. The continuous limit of the saddle-point equations is thus expressed as a singular integral equation for the density

$$⨍ \mathrm{d}y \, \frac{\rho(y)}{x - y} = \frac{8\pi^2}{\lambda} x, \tag{4.9}$$

where the crossed integral sign signifies the principal value. The kernel $K(x - y) = \frac{1}{x-y}$ is the Hilbert kernel and the equation has the famous Wigner semi-

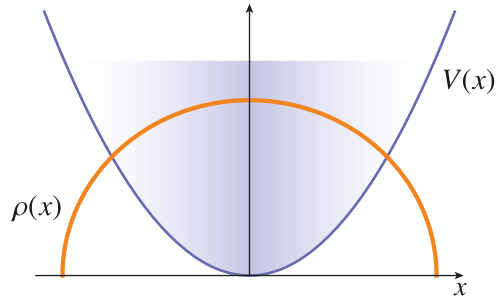


Figure 4.1. The Wigner semi-circle is the Gaussian unitary matrix model solution for the eigenvalue probability distribution $\rho(x)$ plotted here together with the quadratic potential $V(x)$. The faint gradient is a loose schematic illustration of the particle density and filling in the gas interpretation.

circle solution

$$\rho(x) = \frac{2}{\pi\mu}\sqrt{\mu^2 - x^2}\,, \qquad \text{with} \qquad \mu = \frac{\sqrt{\lambda}}{2\pi}\,. \tag{4.10}$$

It is plotted together with a schematic sketch of the potential and its "gas filling" in figure 4.1.

This is only an example and there exists a multitude of different matrix models. However, already this simple one has great relevance for both $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2^*$ SYM, as we will now see.

4.3 Matrix model for $\mathcal{N} = 2^*$ SYM

To employ the powers of localization, we place $\mathcal{N} = 2^*$ SYM on the hypersphere S^4 with radius R. The localization procedure, for both $\mathcal{N} = 2^*$ SYM and its massless limit $\mathcal{N} = 4$ SYM, brings the partition function into one locus, the Coulomb branch moduli space of vacua parametrized by the expectation values of the vector multiplet scalar Y,

$$\langle Y \rangle = \mathrm{diag}(a_1, a_2, \cdots, a_N)\,, \qquad a_i \in \mathbb{R}\,. \tag{4.11}$$

The result is

$$Z = \int \mathrm{d}^N a \prod_{i<j} (a_i - a_j)^2 \, Z_{\text{1-loop}}(a) \, |Z_{\text{inst}}(a)|^2 \, \mathrm{e}^{-\frac{8\pi^2 R^2}{\lambda} N \sum_i a_i^2} \tag{4.12}$$

and their corresponding one-loop determinants are

$$\mathcal{N} = 4 \text{ SYM:} \qquad Z_{\text{1-loop}} = 1\,, \tag{4.13}$$

$$\mathcal{N} = 2^* \text{ SYM:} \qquad Z_{\text{1-loop}} = \prod_{i<j} \frac{H^2(a_i - a_j)}{H(a_i - a_j + M)\, H(a_i - a_j - M)}\,, \tag{4.14}$$

with

$$H(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2 R^2}{n^2}\right) e^{-\frac{x^2 R^2}{n}}. \tag{4.15}$$

The instanton contribution $|Z_{\text{inst}}|$ is 1 for $\mathcal{N} = 4$ SYM but although non-trivial for $\mathcal{N} = 2^*$, it is exponentially suppressed in the large N limit such that it will play no role within our scope. [46–48]

It is now clear that $\mathcal{N} = 4$ SYM in the large N limit is governed by the Gaussian unitary matrix model and that the eigenvalue distribution is precisely the Wigner semi-circle in equation (4.10). Its width is controlled by the 't Hooft coupling and grows smoothly with it, all the way from weak to strong coupling.

The story is very different for $\mathcal{N} = 2^*$ SYM. The saddle-point equations are altered by $Z_{1\text{-loop}}$ which gives the kernel

$$K(x) = \frac{1}{R^2}\frac{1}{x} - \tilde{K}(x) + \frac{1}{2}\tilde{K}(x+M) + \frac{1}{2}\tilde{K}(x-M), \tag{4.16}$$

where

$$\tilde{K}(x) = -\frac{H'(x)}{R^2\, H(x)} = 2x \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{n}{n^2 + x^2 R^2}\right). \tag{4.17}$$

The equation for the density keeps the same right hand side,

$$⨍_{-\mu}^{\mu} \mathrm{d}y\, K(x-y)\rho(y) = \frac{8\pi^2}{\lambda} x, \tag{4.18}$$

where the eigenvalues are still assumed to condense into one interval. This is motivated by the particle interpretation of the eigenvalues as having some interaction force from the kernel but being subjected to the quadratic potential. At small coupling, the potential force is very strong and confines the eigenvalues to the potential well. Gradually increasing λ allows them to spread through their repulsive forces but does not break the distribution into separate intervals.

Despite this now complicated equation, there is still a semi-circle lurking in the background since in the strict strong coupling limit, $\mu \sim \sqrt{\lambda} \to \infty$, the solution (4.10) reappears with only a shift of the endpoints:

$$\rho(x) = \frac{2}{\pi\mu}\sqrt{\mu^2 - x^2}, \qquad \mu = \frac{\sqrt{\lambda(1 + (MR)^2)}}{2\pi}, \qquad (\lambda \to \infty). \tag{4.19}$$

The reason is that the $\tilde{K}$-terms in the kernel $K(x)$ can be approximated as a second order derivative in this limit,

$$\frac{1}{2}\tilde{K}(x-y+M) + \frac{1}{2}\tilde{K}(x-y-M) - \tilde{K}(x-y) \approx \frac{1}{2}\tilde{K}''(x-y)M^2 \approx \frac{M^2}{x-y}, \tag{4.20}$$

since most eigenvalues are on a distance $|x-y| \sim \mu \gg M$. The full kernel then becomes the Hilbert kernel of equation (4.9) with the shifted numerator $1+(MR)^2$.

A more detailed study, though, reveals a very rich structure in $\mathcal{N}=2^*$ SYM; the theory undergoes an infinite number of phase transitions with increasing coupling.

This was first observed in [49] where the integral equation was studied in the decompactification limit $R \to \infty$. An analytical solution was found for ρ for a coupling up to the first critical value $\lambda^{(1)} \approx 35.4$, corresponding to $\mu(\lambda_c^{(1)}) = \frac{M}{2}$. The phase transitions with λ as order parameter can be explained by new very light excitations in the hypermultiplet that enter the spectrum each time there is a resonance

$$M_{ij} = |a_i - a_j \pm M| \approx 0\,. \tag{4.21}$$

The analytical solution of [49] cannot be extended into the strong phase but numerics and subsequent analysis has painted the following picture.

- For $\lambda \leq \lambda_c^{(1)}$, the density has a similar shape to $(\pi\sqrt{\mu^2-x^2})^{-1}$, with the inverse square-root singularities at the boundaries. This is also in accordance with results for pure $\mathcal{N}=2$ SYM, obtained from the very large mass limit. For finite R, the density has square-root behavior at the endpoints $\pm\mu$ but with peaks that grow as R increases. In the decompactification limit, the peaks are not resolved but turn into the inverse square-root singularities.
- At $\lambda = \lambda_c^{(1)}$, the first phase transition occurs and two cusps appear at both ends of ρ at $\pm\mu = (M)/2$. This is a forth order transition.
- As λ increases beyond $\lambda_c^{(1)}$, the amplitudes of the cusps grow and they move towards $x=0$ as more and more pairs of eigenvalues contribute to the resonance (4.21).
- A second phase transition occurs at $\lambda_c^{(2)} \approx 83$ when the width of the eigenvalue distribution hits $2M$. The two cusps of the former transition meet at $x=0$ and two new appear at the endpoints.
- The pattern keeps repeating itself into an ever increasing number of transitions and corresponding cusps as λ increases. The critical values approach $\lambda_c^{(n)} \approx \pi^2 n^2$ for asymptotically large λ while the underlying shape of the distribution is shifting from convex to concave.
- In the strict strong coupling limit, there is an infinite number of phase transitions and the eigenvalue distribution averages out to the Wigner semi-circle (4.19).

Several contributions have since been made to the incremental understanding of this theory. An endpoint analysis was done in [50] while an ansatz for the strong-coupling phase was proposed in [51]. A wider scope of theories

were considered in [52, 53] making connection to the Seiberg-Witten curve. [54] studied Wilson loops of higer rank symmetric and anti-symmetric representations which was compared to a D-brane construction in the holographic dual in [55]. The first corrections to the effective string tension in the Pilch-Warner background could be computed in [41].

Paper III aimed to put the ansatz for the strong phase of [51] on firmer ground, making connections to the holographic dual, and to compute the next-to-next-to-leading order corrections to the free energy and susceptibility. The result showed why no phase transition has yet been observed in the holographic dual since it occurs in the NNLO for which the string corrections have not been computed. It is, however, of perturbative origin so a direct comparison should be possible.

5. Spin-chains and integrability

Spin-chains were the first systems found with quantum integrability and have been an active field of research ever since. The first break-through came with Bethe's ansatz for the Heisenberg model in the early 1930's onto which several major developments have been built. Baxter was responsible for many of these during the 70's and the algebraic Bethe ansatz came out of the quantum inverse scattering method of the Leningrad-school centered around Faddeev.

This chapter introduces models and formalisms from these historic developments relevant for the applications in the appended papers. It is aimed to be self-contained and draws heavily from many of earlier expositions of the subject, above all from [56]. For other examples of good introductory references, see [57–60]. The section about Q-functions and Q-operators is more or less condensed parts of [61].

5.1 The Heisenberg $XXX_{1/2}$ spin-chain

The Heisenberg spin-chain was proposed in the early days of quantum mechanics as a simple quantum model for magnetism. It models a one-dimensional lattice of particles whose only interactions are through their spin. We will begin with restricting to spin-1/2 particles.

The lattice can have different boundary conditions but the most studied spin-chains are (semi-)periodic; a schematic illustration of such a chain can be found in figure 5.1. The periodic chains will dominate in this thesis and we will only make a few remarks on chains with open boundary conditions.

The Hilbert space for a spin-chain of length L is a tensor product of the local Hilbert spaces $\mathcal{H}_l$ at each site,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_L \,. \tag{5.1}$$

Since we are looking at a chain of spin-1/2 particles, the local Hilbert spaces are $\mathcal{H}_l = \mathbb{C}^2$, spanned by the two vectors $|\uparrow\rangle$ and $|\downarrow\rangle$. They can be acted upon by the standard spin matrices

$$S^i = \tfrac{1}{2}\sigma_i \,, \tag{5.2}$$

expressed in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.3}$$

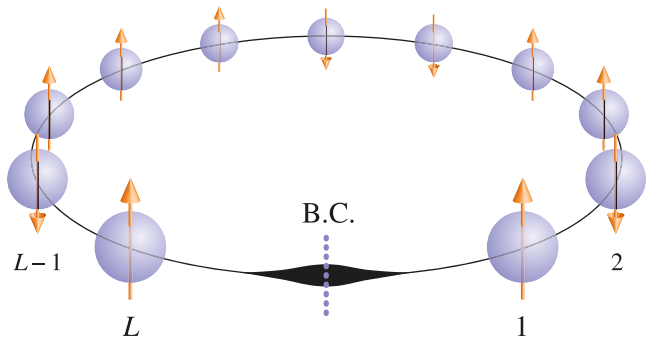


Figure 5.1. The semi-periodic Heisenberg-chain is a one-dimensional lattice of L particles that interact with their nearest neighbors through spin-spin interactions. The boundary conditions (B.C.) can be either fully periodic or contain some sort of twist. We will study twisted boundary conditions in a later section of this chapter (sec. 5.5).

A state vector in the full Hilbert space can be written as

$$|\uparrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\uparrow\rangle = |\uparrow\downarrow \cdots \uparrow\rangle \in \mathcal{H} \tag{5.4}$$

and the local action of a spin matrix on such a state at position l is denoted

$$S^i_l = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \underbrace{\otimes S^i \otimes}_{\text{site } l} \mathbb{1} \otimes \cdots \otimes \mathbb{1}\,. \tag{5.5}$$

The spin matrices for the full Hilbert space $\mathcal{H}$ is the sum of the local action on all sites,

$$S^i = \sum_{l=1}^{L} S^i_l, \tag{5.6}$$

and we denote it with the same symbol as the single space spin-matrices above.

The Hamiltonian for the $XXX_{1/2}$-chain[1] couples the spins at neighboring sites through

$$\begin{aligned} H &= 4\sum_{l=1}^{L}\left(\frac{1}{4} - \vec{S}_l \cdot \vec{S}_{l+1}\right) = 4\sum_{l=1}^{L}\left(\frac{1}{4} - S^3_l S^3_{l+1} - \frac{1}{2}S^+_l S^-_{l+1} - \frac{1}{2}S^-_l S^+_{l+1}\right) \\ &= 2\sum_{l=1}^{L}\left(1 - \mathbb{P}_{l,l+1}\right), \end{aligned} \tag{5.7}$$

where $\mathbb{P}_{l,l+1}$ denotes the permutation of site l and $l+1$. The boundary conditions are perfectly periodic, which means that

$$S^i_{L+1} = S^i_1\,. \tag{5.8}$$

[1]The name XXX comes from the fact that all the spin couplings in each spin-space direction are equal. A more general Hamiltonian can have a different coupling for the $S^3_l S^3_{l+1}$-term, which is called the XXZ-chain. If all directions have different coupling, it is logically named the XYZ-chain. All these models are integrable but they will not be relevant for this thesis.

We will look at a different boundary condition later on.

The Hamiltonian (5.7) enjoys a lot of symmetries. It is both translation and parity invariant and does also satisfy

$$[H, \vec{S}] = 0\,, \tag{5.9}$$

i.e. it has full $SU(2)$-symmetry. This means that the number of down spins, $|\downarrow\rangle$, is conserved and we denote this number as M. We will view the down spins as excitations, called magnons, in a sea of spin-up and write

$$|\Omega\rangle = |\uparrow\uparrow\cdots\uparrow\rangle \tag{5.10}$$

as a reference state, also referred to as the spin-chain vacuum. Note that it has energy eigenvalue zero.

These are the manifest symmetries but the Hamiltonian (5.7) does in fact have *a lot* more symmetries. By virtue of them, it is possible to construct the energy eigenstates and hence diagonalize the entire Hamiltonian. The Heisenberg spin-chain is hence a solvable model and the first and most studied among integrable lattice models.

Let us look at two very different methods of accomplishing this diagonalization which in the end, as they should, yield the same result.

5.2 Coordinate Bethe Ansatz

The first approach is the coordinate Bethe ansatz (CBA) which was proposed by Hans Bethe already in 1931 [62]. It assumes the wavefunction to be a spin-wave, an eigenstate to translations, of M magnons that is parametrized by M (pseudo)-momenta $\mathbf{p} = (p_1, p_2, \cdots, p_M)$. The ansatz reads

$$|\Psi\rangle = \Psi^{s_1,s_2,\ldots,s_L}|s_1 s_2 \cdots s_L\rangle = \sum_{\sigma\in S_M} \sum_{1\le n_1<\ldots<n_M\le M} \mathrm{e}^{\mathrm{i}p_{\sigma_i}n_i} S_\sigma(\mathbf{p})|n_1, n_2, \cdots, n_M\rangle\,, \tag{5.11}$$

where s_l runs over $\uparrow,\downarrow$, repeated indices are summed over and $|n_1,\cdots,n_M\rangle$ is a state vector with down spins at sites $n_1, n_2, \cdots, n_M$. We will sometimes write the collective labels $|\mathbf{s}\rangle$ and $|\mathbf{n}\rangle$ for the ket-notations in (5.11).

σ denotes elements in the permutation group S_M and the final object in the ansatz, S_σ, is very important. It is the product of S-matrices corresponding to the permutation σ, e.g.

$$S_{1432} = S_{34}S_{24}S_{23}\,, \tag{5.12}$$

where a single S-matrix governs the scattering when two excitations pass each other,

$$S_{ij} = -\frac{1 + \mathrm{e}^{\mathrm{i}(p_i+p_j)} - 2\,\mathrm{e}^{\mathrm{i}p_j}}{1 + \mathrm{e}^{\mathrm{i}(p_i+p_j)} - 2\,\mathrm{e}^{\mathrm{i}p_i}}\,. \tag{5.13}$$

For equation (5.12) to be well-defined, the S-matrices must satisfy the compatibility condition of two equivalent permutations:

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}\,, \tag{5.14}$$

which obviously is the case here.

The Bethe state $|\Psi\rangle$ is an eigenstate to the translation operator $\hat{U}$ by construction with the eigenvalue being

$$\hat{U}|\Psi\rangle = \mathrm{e}^{-\mathrm{i}\sum_i p_i}|\Psi\rangle\,, \tag{5.15}$$

where

$$\hat{U}|i_1, i_2, \ldots, i_L\rangle = |i_L, i_1, \ldots, i_{L-1}\rangle\,. \tag{5.16}$$

Hence, the total momentum of a Bethe state is

$$p_{\text{tot}} = \sum_{i=1}^{M} p_i\,. \tag{5.17}$$

Since we in this thesis employ the spin-chain picture to compute quantities related to single-trace operators in field theories, we will need to impose the zero momentum condition

$$\sum_{i=1}^{M} p_i = 0 \tag{5.18}$$

to consider translationally invariant states, thus mirroring the cyclicity of the trace.

It is often convenient to instead of the momenta parametrize the ansatz through the variables

$$u_i = \frac{1}{2}\cot\frac{p_i}{2}\,, \tag{5.19}$$

called rapidities, such that

$$\mathrm{e}^{\mathrm{i}p_i} = \frac{u_i + \frac{\mathrm{i}}{2}}{u - \frac{\mathrm{i}}{2}}\,, \qquad\qquad S_{ij} = \frac{u_i - u_j - \mathrm{i}}{u_i - u_j + \mathrm{i}}\,. \tag{5.20}$$

We will use both notations, choosing which ever is the most practical for the current purpose.

5.2.1 The Bethe equations

The goal with the ansatz $|\Psi\rangle$ was to get eigenstates to the Hamiltonian but acting with H on $|\Psi\rangle$ does not immediately confirm this. A direct computation

shows that the momenta (or rapidities) must satisfy the non-linear equation system

$$e^{ip_k L} \prod_{j \neq k} S_{kj} = 1\,, \qquad (k = 1, 2, \ldots, M) \tag{5.21}$$

or equivalently in rapidities

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L \prod_{j \neq k} \frac{u_k - u_j - i}{u_k - u_j + i} = 1\,. \tag{5.22}$$

These are called the Bethe equations and a solution to them is a set of Bethe roots. They guarantee that the ansatz indeed produces an eigenstate and they determine the energy eigenvalue through

$$E = 2 \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}} = 8 \sum_{k=1}^{M} \sin^2 \frac{p_k}{2}\,. \tag{5.23}$$

The Bethe equations are, however, hard to solve in practice. There are many solutions and not all of them correspond to physical spin-chain states, e.g. only solutions with distinct[2] roots are physical solutions. Note also that the Bethe roots may be complex.

Many methods have been devised to solve the Bethe equations and we will discuss a highly efficient method, developed in [63], in chapter 7. Another recent method connects the Bethe equations with algebraic geometry but we will not follow that approach here [64].

5.2.2 The spectrum

A curious fact of the Bethe equations (5.22) is that we can add a rapidity u_{M+1} sitting at infinity and still have a solution. This is related to the $SU(2)$-symmetry of the model. Since the spin-operators S^i commute with the Hamiltonian we can lower the spin of an eigenstate

$$(S^-)^n |\Psi\rangle \tag{5.24}$$

and still have an eigenstate with the same energy, as long as n is not large enough to annihilate the state. This is the same as adding a magnon with zero momentum.

Bethe states on the form (5.24) are called Bethe descendants and the total spectrum arranges itself according to the irreducible decomposition of the L-tensor representation of $SU(2)$. Each (physical) solution to the Bethe equations corresponds to a highest weight state in this decomposition. The spectrum is

[2] Strictly speaking, this holds for finite roots. There is an exception for zero momenta/infinite rapidities discussed below.

Table 5.1. *Reference table for the different operators in the algebraic Bethe ansatz.*

Name	Symbol	Acts in
Lax operator	$\mathcal{L}_{a,l}(u)$	$\mathcal{V}_a \otimes \mathcal{H}_l$
Monodromy matrix	$\mathcal{M}_a(u)$	$\mathcal{V}_a \otimes \mathcal{H}$
R-matrix	$\mathcal{R}_{12}(u,v)$	$\mathcal{V}_{a_1} \otimes \mathcal{V}_{a_2}$
Transfer matrix	$\mathcal{T}(u)$	$\mathcal{H}$
Creation operator	$\mathrm{B}(u)$	$\mathcal{H}$

thus highly degenerate. It also means that we need to have $M \leq L/2$ in order to have a meaningful description.

The local Hamiltonian in the summand of equation (5.7) can in this spirit be viewed as the projection onto the triplet in the decomposition of $\mathcal{H}_l \otimes \mathcal{H}_{l+1}$.

5.2.3 Norm of Bethe states

The norm of a Bethe state in the coordinate ansatz can be written through a determinant of its rapidities in a formula called the Guadin norm [65, 66]. It is expressed in terms of the logarithm of the left hand side of the Bethe equations (5.22), such that they read

$$\mathrm{e}^{\mathrm{i}\Phi_k} = 1 \tag{5.25}$$

The Gaudin norm is then

$$\langle \Psi | \Psi \rangle = \prod_{k=1}^{M} (u_k^2 + \tfrac{1}{4}) \det_{i,j} \frac{\partial \Phi_i}{\partial u_j}. \tag{5.26}$$

Assuming that we have a good way of solving the Bethe equations, we have now diagonalized the Hamiltonian with access to both eigenstates and their corresponding eigenvalues. But we have not yet seen the full amount of symmetry that the system possesses. We hence move to a method in which this is much more apparent.

5.3 Algebraic Bethe Ansatz

The coordinate Bethe ansatz is very direct in its physical interpretation. That is somewhat obscured in the Algebraic Bethe Ansatz (ABA) but that is the price we pay for transparency of the integrable structure. The formalism is built on a number of objects and for clarity they are listed in table 5.1, provided as a reference for when they are consecutively introduced in the text.

The overall goal is still to diagonalize the spin-chain Hamiltonian (5.7) but with control over all the symmetries in the theory. We denote the infinite

number of charges Q_n, the Hamiltonian being one of them, and since they are all symmetries we have

$$[Q_n, Q_m] = 0\,. \tag{5.27}$$

We can collect them in an operator valued function

$$\mathcal{T}(u) = \exp\left(\mathrm{i}\sum_n \frac{(u-\frac{\mathrm{i}}{2})^n}{n!} Q_n\right) \tag{5.28}$$

called the transfer matrix. The variable u is called the spectral parameter and we can ensure the commutativity (5.27) of the entire tower of charges by requiring

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0 \tag{5.29}$$

for any two values u, v of the spectral parameters. We will now show how to construct this transfer matrix.

The first step is to introduce what is called an auxiliary space $\mathcal{V}_a$; it can in principle be any vector space but we will start by setting $\mathcal{V}_a = \mathbb{C}^2$, as is the case for the local Hilbert spaces $\mathcal{H}_l$. It is purely a mathematical tool with no physical meaning but it is central to the construction. Each auxiliary space comes with a spectral parameter associated with it.

The basic building block for the ABA is the Lax operator $\mathcal{L}_{al}(u)$, which acts in the tensor product space $\mathcal{V}_a \otimes \mathcal{H}_l$ and depends on the spectral parameter u corresponding to $\mathcal{V}_a$. It reads

$$\mathcal{L}_{a,l}(u) = u\mathcal{I} + \mathrm{i}\sigma_i \otimes S^i = (u - \frac{\mathrm{i}}{2})\mathcal{I} + \mathrm{i}\mathcal{P}\,, \tag{5.30}$$

where $\mathcal{I}$ is the identity operator and $\mathcal{P}$ is the permutation operator

$$\mathcal{P}(a \otimes b) = b \otimes a\,. \tag{5.31}$$

Written as a matrix in the auxiliary space, with elements that are operators in $\mathcal{H}_l$, the Lax operator can also be written as

$$\mathcal{L}_{a,l}(u) = \begin{pmatrix} u + \mathrm{i}S^3_l & \mathrm{i}S^-_l \\ \mathrm{i}S^+_l & u - \mathrm{i}S^3_l \end{pmatrix}. \tag{5.32}$$

We can multiply a number of Lax operators that share the same auxiliary space, and hence spectral parameter, into

$$\mathcal{M}_a(u) = \mathcal{L}_{a,L}(u)\mathcal{L}_{a,L-1}(u)\ldots\mathcal{L}_{a,1}(u)\,. \tag{5.33}$$

The resulting object $\mathcal{M}_a(u)$ is called the monodromy matrix and taking the trace over the auxiliary space, we finally obtain the transfer matrix

$$\mathcal{T}(u) = \mathrm{tr}_a\, \mathcal{M}_a(u)\,. \tag{5.34}$$

To ensure the commutation relation (5.29) we need to have

$$\mathrm{tr}_{a_1\otimes a_2}\,\mathcal{M}_{a_1}(u)\mathcal{M}_{a_2}(v) = \mathrm{tr}_{a_1\otimes a_2}\,\mathcal{M}_{a_2}(v)\mathcal{M}_{a_1}(u)\,. \tag{5.35}$$

This is warranted by the existence of an intertwining matrix, called the R-matrix, which acts in the tensor product of two auxiliary spaces $\mathcal{V}_{a_1}\otimes\mathcal{V}_{a_2}$ such that

$$\mathcal{R}_{12}(u,v)\mathcal{M}_1(u)\mathcal{M}_2(v) = \mathcal{M}_2(v)\mathcal{M}_1(u)\mathcal{R}_{12}(u,v)\,. \tag{5.36}$$

Here we have abbreviated the auxiliary space subscripts to only their numbers which we will mostly do as long as it does not create confusion with the enumerating of the physical local Hilbert spaces.

However, if we want to relate the product of three monodromy matrices $\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3$ to $\mathcal{M}_3\mathcal{M}_2\mathcal{M}_1$ we can do that in two different ways, just as for the S-matrices in the coordinate ansatz. The consistency condition on the R-matrix is

$$\mathcal{R}_{12}(u,v)\mathcal{R}_{13}(u,w)\mathcal{R}_{23}(v,w) = \mathcal{R}_{23}(v,w)\mathcal{R}_{13}(u,w)\mathcal{R}_{12}(u,v) \tag{5.37}$$

and it is called the Yang-Baxter equation. The existence of an R-matrix that obeys the Yang-Baxter equation is fundamental to quantum integrable systems and is what underlies the commuting tower of charges.

An explicit solution to the R-matrix is

$$\mathcal{R}_{a_1a_2}(u,v) = \mathcal{R}_{a_1a_2}(u-v) = (u-v)\mathcal{I} + \mathrm{i}\mathcal{P}\,, \tag{5.38}$$

which we observe is the same as the Lax operator when $\mathcal{H}_l = \mathcal{V} = \mathbb{C}^2$, just with a shifted spectral parameter.

We now have a construction which lay bare the integrable structure and we have access to all of the commuting charges via

$$Q_{n+1} = -\mathrm{i}\frac{\mathrm{d}^n}{\mathrm{d}u^n}\log\mathcal{T}(u)\Big|_{u=\frac{\mathrm{i}}{2}}\,, \qquad n\geq 0\,. \tag{5.39}$$

Among these, we have the momentum operator which is related to Q_1 since $\mathcal{T}(\frac{\mathrm{i}}{2})$ is proportional to the translation operator, while Q_2 is the spin-chain Hamiltonian (5.7) (up to normalization and a shift).

To fully solve the model, however, we need the simultaneous eigenstates to all of these charges.

5.3.1 Bethe states in the ABA

The monodromy matrix can be seen as a matrix in the auxiliary space with entries that are operators in the full physical space $\mathcal{H}$,

$$\mathcal{M}(u) = \begin{pmatrix} \mathrm{A}(u) & \mathrm{B}(u) \\ \mathrm{C}(u) & \mathrm{D}(u) \end{pmatrix}. \tag{5.40}$$

The relation (5.36) determines the commutation relation between these operators and with some algebra it can be shown that $\mathrm{B}(u)$ acts as a creation operator and $\mathrm{C}(u)$ as an annihilation operator on the reference state $|\Omega\rangle$. Moreover, the state

$$|\mathbf{u}\rangle = \mathrm{B}(u_1)\mathrm{B}(u_2)\dots\mathrm{B}(u_M)|\Omega\rangle \tag{5.41}$$

is an eigenstate to the transfer matrix provided that the set of M spectral parameters $\mathbf{u}$ satisfy a condition. Again, this turns out to be the Bethe equations (5.22). The calculation is a bit lengthy but can be found in detail in [56].

The transfer matrix eigenvalue of $|\mathbf{u}\rangle$ is

$$\begin{gathered}\mathcal{T}(u)|\mathbf{u}\rangle = \Lambda(u|\mathbf{u})|\mathbf{u}\rangle\,,\\ \Lambda(u|\mathbf{u}) = (u+\frac{\mathrm{i}}{2})^L\prod_{k=1}^{M}\frac{u-u_k-\mathrm{i}}{u-u_k} + (u-\frac{\mathrm{i}}{2})^L\prod_{k=1}^{M}\frac{u-u_k+\mathrm{i}}{u-u_k}\,,\end{gathered} \tag{5.42}$$

and we stress again that $\mathbf{u}$ must satisfy the Bethe equations (5.22) in order for this to hold.

The eigenstates $|\mathbf{u}\rangle$ in the algebraic Bethe ansatz are proportional to the Bethe states in the coordinate ansatz but not identical.

5.3.2 A note on open boundary conditions

Although we exclusively deal with closed spin-chains in this thesis, there is also an ample literature on integrable spin-chains with open boundary conditions. The algebraic Bethe ansatz for these chains is very similar. It accounts for the boundaries by double-row transfer matrices in which two product chains of Lax operators are included, but with different orderings:

$$\tilde{\mathcal{T}}_a = \mathrm{tr}_a\Big(\tilde{K}_a(u)\ \mathcal{L}_{a,L}(u)\mathcal{L}_{a,L-1}(u)\dots\mathcal{L}_{a,1}(u)\ K_a(u)\ \mathcal{L}_{1,a}(u)\mathcal{L}_{2,a}(u)\dots\mathcal{L}_{L,a}(u)\Big)\,. \tag{5.43}$$

The matrices K_a and $\tilde{K}_a$ are called reflection matrices and describe the actual boundary conditions. They have to satisfy the reflection equation, also called the boundary Yang-Baxter equation,

$$\mathcal{R}_{ab}(u-v)K_a(u)\mathcal{R}_{ba}(u+v)K_b(v) = K_b(v)\mathcal{R}_{ab}(u+v)K_a(u)\mathcal{R}_{ba}(u-v) \tag{5.44}$$

in order for the double-row transfer matrices $\tilde{\mathcal{T}}$ to commute and hence preserve integrability.

We will not delve deeper into the subject of open spin-chains but instead move on with these comments only as a reference for the next topic.

5.4 Matrix Product States

There exists a certain class of states in the spin-chain Hilbert space called Matrix Product States (MPS). They are of the form

$$|\mathrm{MPS}\rangle = \mathrm{tr}\left(A_1^{(i_1)} A_2^{(i_2)} \dots A_L^{(i_L)}\right) |i_1, i_2, \dots, i_L\rangle\,, \tag{5.45}$$

where the spin-site indices i_l run over all dimensions in $\mathcal{H}_l$ (i.e. over $\uparrow, \downarrow$ in our present case) and $A_l^{(i_l)}$ are matrices of dimension $d_l \times d_{l+1}$, where d_l are called the bond dimensions. In fact, all states in $\mathcal{H}$ can be written on this form for sufficiently large bond dimensions so the term MPS normally refers to the subset of such states for which d_l do not scale with the system size L [67].

These states will become relevant for one-point functions in $\mathcal{N} = 4$ SYM with a defect, studied below in chapter 3.

MPSs are common in tensor networks and have been used as initial states in quantum quenches. In fairly recent work, [68] studied what type of an MPS is integrable.

The theories considered are one-dimensional lattice models in which the integrability tower of commuting charges obeys the reflection property

$$\Pi Q_n \Pi = (-1)^n Q_n\,, \tag{5.46}$$

where Π is the parity operator. The MPS is furthermore required to be p-periodic, i.e.

$$\hat{U}^p |\mathrm{MPS}\rangle = |\mathrm{MPS}\rangle\,. \tag{5.47}$$

The MPSs we will encounter in this thesis have all bond dimensions being equal and are translationally invariant.

The proposed definition for an integrable MPS under these conditions is that it is annihilated by all the odd charges:

$$Q_{2n+1}|\mathrm{MPS}\rangle \stackrel{!}{=} 0\,, \qquad n \geq 1\,. \tag{5.48}$$

Equivalently, it can be phrased as

$$\begin{cases} \langle \mathrm{MPS}|\hat{U}^{p-2}\mathcal{T}(u)\mathcal{T}(-u)|\mathrm{MPS}\rangle \stackrel{!}{=} 1 + \mathcal{O}(u^L)\,, \\ \langle \mathrm{MPS}|\Pi\hat{U}^{p-2}\mathcal{T}(u)\mathcal{T}(-u)\Pi|\mathrm{MPS}\rangle \stackrel{!}{=} 1 + \mathcal{O}(u^L)\,, \end{cases} \tag{5.49}$$

for small u.

The condition (5.48) is implied by the stronger requirement

$$\Pi\mathcal{T}(u)\Pi|\mathrm{MPS}\rangle = \mathcal{T}(u)|\mathrm{MPS}\rangle\,, \tag{5.50}$$

which also implies a p-periodicity of 2. We will mainly use this condition below.

5.4.1 A note on MPSs from reflection matrices

It was shown in [68] that, under certain conditions on the R-matrix, an integrable MPS can be constructed from a type of reflection matrices satisfying equation (5.44). A similar approach was used in [69] where the MPS arising in the setting of $\mathcal{N} = 4$ SYM with a defect could be reconstructed from partial Néel states[3] through the use of non-diagonal reflection matrices, providing integrability methods for calculating MPS-overlaps with Bethe states.

It would be interesting to investigate further what of these approaches could be adapted to the twisted Heisenberg chain, which we now will introduce.

5.5 Heisenberg $XXX_{1/2}$ with twisted boundary condition

Let us now return the question about the boundary conditions and modify them slightly. Before gluing together the two ends of the chain, we twist them by inserting a phase β. This twist parameter is the correspondence in the spin-chain picture to the deformation parameter in the β-deformed $\mathcal{N} = 4$ SYM theory in section 3.1. We can implement it by changing the identification (5.8) into

$$S^{\pm}_{L+1} = \mathrm{e}^{\mp \mathrm{i}\beta L} S^{\pm}_{1}, \qquad S^{3}_{L+1} = S^{3}_{1}. \tag{5.51}$$

Apart from this change, the Hamiltonian keeps the form the expression (5.7),

$$\tilde{H} = 4 \sum_{l=1}^{L} \left(\frac{1}{4} - S^{3}_{l} S^{3}_{l+1} - \frac{1}{2} S^{+}_{l} S^{-}_{l+1} - \frac{1}{2} S^{-}_{l} S^{+}_{l+1} \right), \tag{5.52}$$

but where we have added the tilde to indicate the now twisted boundary condition.

The resulting system is still an integrable spin-chain but the former $SU(2)$-symmetry is now broken. What remains is the Cartan $U(1)$, with $[\tilde{H}, S^3] = 0$, so the excitation number M is still conserved.

We now move on to diagonalize this twisted spin-chain following closely the recipes of the former sections.

5.5.1 Coordinate Bethe Ansatz

The coordinate Bethe ansatz still works for the twisted boundary conditions, with only some minor adjustments. However, to better reflect the operators in the field theory, we choose to work in another basis. With the similarity

[3] The Néel state is the state with alternating spin-up and spin-down and the Mth partial Néel state is the state with M spin-down distributed in all ways such that there always is an odd number of spin-up in between each pair of spin-down.

transformation

$$C(\beta) = \mathrm{e}^{-\mathrm{i}\beta L S^3} \otimes \mathrm{e}^{-\mathrm{i}\beta(L-1)S^3} \otimes \cdots \otimes \mathrm{e}^{-\mathrm{i}\beta S^3}, \tag{5.53}$$

we can write the Hamiltonian in the form

$$H_\beta = C(\beta)\tilde{H}C(\beta)^{-1}, \tag{5.54}$$

$$H_\beta = 4\sum_{l=1}^{L}\left(\frac{1}{4} - S_l^3 S_{l+1}^3 - \frac{1}{2}\mathrm{e}^{\mathrm{i}\beta} S_l^+ S_{l+1}^- - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\beta} S_l^- S_{l+1}^+\right). \tag{5.55}$$

The point of this is that the boundary conditions again are periodic as in equation (5.8) which makes the translations and cyclic states behave just as before. Although it is more customary in the study of integrable spin-chains to keep the twist in the boundary condition, we work with this basis since the map to single-trace operators is more direct.

The Hamiltonian H_β is diagonalized by the ansatz

$$|\Psi\rangle = \sum_{\sigma\in S^M} \sum_{1\le n_1<\ldots<n_M\le M} \mathrm{e}^{\mathrm{i}(p_{\sigma_i}+\beta)n_i} S_\sigma(\mathbf{p})|n_1, n_2, \cdots, n_M\rangle, \tag{5.56}$$

if the momenta satisfy the now twisted Bethe equations

$$\mathrm{e}^{\mathrm{i}p_k L}\mathrm{e}^{\mathrm{i}\beta L}\prod_{j\neq k} S_{kj} = 1, \qquad \left(\frac{u_k + \mathrm{i}/2}{u_k - \mathrm{i}/2}\right)^L \mathrm{e}^{\mathrm{i}\beta L}\prod_{j\neq k}\frac{u_k - u_j - \mathrm{i}}{u_k - u_j + \mathrm{i}} = 1. \tag{5.57}$$

The S-matrix stays the same, i.e. it is still given by equation (5.13) (or (5.20)).

Note, however, that the expression for the total momentum is changed. The translation eigenvalue is no longer related only to the sum of momenta but rather to

$$p_{\mathrm{tot}} = \sum_{k=1}^{M}(p_k + \beta) \tag{5.58}$$

since $\hat{U}$ acts just as before. The zero momentum condition from the trace cyclicity does hence involve a term with the twist.

5.5.2 The spectrum

The broken $SU(2)$-symmetry lifts all the degeneracy such that every solution to the twisted Bethe equations gives a state with a distinct energy eigenvalue, still given by

$$E = 2\sum_{k=1}^{M}\frac{1}{u_k^2 + \frac{1}{4}}. \tag{5.59}$$

The twist does only enter the energy implicitly through the Bethe roots.

Another important aspect of the lifted spectrum is that it simultaneously holds a dual spin-chain. Just as well as we consider the M spin-downs as excitations, we could view the $L-M$ spin-ups as excitations in a sea of M spin-downs. The corresponding Bethe equations are identical but with a sign flip for the twist β, where there is now a new set of $L-M$ dual Bethe roots $v_k \in \mathbf{v}$. We will return to this feature in section 5.6 in the context of Q-functions.

5.5.3 Algebraic Bethe ansatz

The twist enters the algebraic Bethe ansatz as a constant twist matrix κ in the auxiliary space. Specifically, the monodromy matrix becomes

$$\mathcal{M}_a(u) = \begin{pmatrix} e^{i\beta L/2} & 0 \\ 0 & e^{-i\beta L/2} \end{pmatrix} \mathcal{L}_{a,L}(u) \dots \mathcal{L}_{a,1}(u) . \tag{5.60}$$

This retains the integrable structure as long as

$$[\kappa_1 \otimes \kappa_2, \mathcal{R}_{12}] = 0 \tag{5.61}$$

since that ensures that the equation (5.36) still holds and that in turn the twisted transfer matrix

$$\mathcal{T}(u) = \mathrm{tr}_a \left[\begin{pmatrix} e^{i\beta L/2} & 0 \\ 0 & e^{-i\beta L/2} \end{pmatrix} \mathcal{L}_{a,L}(u) \dots \mathcal{L}_{a,1}(u) \right] \tag{5.62}$$

still commutes at different values of the spectral parameter.

The R-matrix in (5.38) is in fact $GL(2)$-invariant such that it satisfies equation (5.61) for any (2×2)-dimensional matrix κ. We can thus clearly see that the insertion of the diagonal matrix in the monodromy matrix (5.60) is compatible with integrability.

Everything from the periodic ABA in section 5.3 goes through with only the modification of the Bethe equations into the twisted version in (5.57). This construction yields the Hamiltonian $\tilde{H}$ with the twisted boundary conditions.

If we would like to obtain H_β directly from the logarithmic derivative of the transfer matrix, we could instead start with the twisted Lax operator

$$\mathcal{L}_{a,l}(u) = (u - \tfrac{i}{2}) e^{-i\mathbf{q}_a \wedge \mathbf{q}_l} \mathcal{I}_{al} + i\mathcal{P}_{al} , \tag{5.63}$$

where $\mathbf{q}_a \wedge \mathbf{q}_l$ is a truncated version of (3.2) to our two-dimensional space. The transfer matrix is then built from a monodromy matrix without the twist matrix κ.

5.6 Q-functions

There is a third, and perhaps the most elegant, way of obtaining the spin-chain spectrum and it goes through what is called Q-functions and Q-operators.

Consider the Bethe equations (5.57). We can reformulate them in terms of the Q-polynomials, corresponding to each eigenstate,

$$A_-(u) = \prod_{k=1}^{M}(u-u_k)\,, \qquad\qquad A_+(u) = \prod_{k=1}^{L-M}(u-v_k)\,, \tag{5.64}$$

such that

$$\left(\frac{u+\frac{\mathrm{i}}{2}}{u-\frac{\mathrm{i}}{2}}\right)^L \mathrm{e}^{\pm \mathrm{i}\beta L} = -\frac{A_\mp(u+\mathrm{i})}{A_\mp(u-\mathrm{i})}\,, \qquad\qquad u\in\mathbf{u},\mathbf{v}\,, \tag{5.65}$$

captures the Bethe equations for both viewpoints on which the direction of the excitations is.

As a consequence, there has to exist another polynomial $T(u)$ such that

$$T(u)A_\mp(u) = \mathrm{e}^{\pm \mathrm{i}\beta L/2}\left(u+\tfrac{\mathrm{i}}{2}\right)^L A_\mp(u-\mathrm{i}) + \mathrm{e}^{\mp \mathrm{i}\beta L/2}\left(u-\tfrac{\mathrm{i}}{2}\right)^L A_\mp(u+\mathrm{i}) \tag{5.66}$$

holds for all eigenstates and all complex values of the spectral parameter u. The right hand side has to be a polynomial proportional to $A_\mp(u)$ since it shares the same roots, $\mathbf{v}$ or $\mathbf{u}$, depending on the sign. That $T(u)$ is independent of this sign is however not apparent but is proved in [61]. In consequence,

$$T(u) = 2\cos\frac{\beta L}{2}\prod_{k=1}^{L}(u-w_k)\,, \tag{5.67}$$

for some roots w_k to match the orders on both sides.

Equation (5.66) can be rewritten as a second order difference equation

$$T(u)Q(u) = \left(u+\tfrac{\mathrm{i}}{2}\right)^L Q(u-\mathrm{i}) + \left(u-\tfrac{\mathrm{i}}{2}\right)^L Q(u+\mathrm{i}) \tag{5.68}$$

for the function $Q(u)$ with the two linearly independent solutions

$$Q_\pm(u) = \mathrm{e}^{\pm u\beta L/2}A_\pm(u)\,. \tag{5.69}$$

These are called Baxter Q-functions, after they were first introduced by Baxter for the 8-vertex model in [70].

Equation (5.68) can be promoted to an operator equation

$$\mathcal{T}(u)\mathcal{Q}(u) = \left(u+\tfrac{\mathrm{i}}{2}\right)^L \mathcal{Q}(u-\mathrm{i}) + \left(u-\tfrac{\mathrm{i}}{2}\right)^L \mathcal{Q}(u+\mathrm{i})\,, \tag{5.70}$$

for the commuting operators $\mathcal{T}$ and $\mathcal{Q}$, such that equation (5.66) holds for their respective eigenvalues. Here, $\mathcal{T}$ is in fact the transfer matrix (5.34).

It is possible to prove the commutativity of $\mathcal{T}$, $\mathcal{Q}_+$ and $\mathcal{Q}_-$ through a generalized Lax operator construction, where the former two-dimensional σ_i in (5.30) are replaced by J_i and the auxiliary space $\mathcal{V}_a$ is taken to be the module for the

j-representation of $\mathfrak{sl}(2)$. j can be the half-integer of the $(2j+1)$-dimensional spin representations but is allowed to take any complex values.

The transfer matrices resulting from a trace of such Lax operators are denoted $\mathcal{T}_j$, or $\mathcal{T}_j^+$ for the infinite dimensional representations. The twist β functions as a regulator in the infinite sums of these traces.

From the construction of $Q_\pm$, written below in equation 5.79, it can be shown that

$$f(\beta)\mathcal{T}_j(u) = Q_+(u+\mathrm{i}j+\tfrac{\mathrm{i}}{2})Q_-(u-\mathrm{i}j-\tfrac{\mathrm{i}}{2}) - Q_-(u+\mathrm{i}j+\tfrac{\mathrm{i}}{2})Q_+(u-\mathrm{i}j-\tfrac{\mathrm{i}}{2}), \tag{5.71}$$

with $f(\beta) = 2\mathrm{i}\sin(\beta L/2)$ and j required to be a half-integer. This immediately implies the two special cases $j = 0$ and $j = \frac{1}{2}$. The former gives

$$f(\beta)u^L = Q_+(u+\tfrac{\mathrm{i}}{2})Q_-(u-\tfrac{\mathrm{i}}{2}) - Q_-(u+\tfrac{\mathrm{i}}{2})Q_+(u-\tfrac{\mathrm{i}}{2}), \tag{5.72}$$

and combining this with the equation for $j = \frac{1}{2}$ yields equation (5.68).

Consider again the case $j = 0$ in equation (5.72). If we define two additional and trivial Q-functions,

$$Q_\emptyset = u^L, \qquad\qquad Q_{12} = 1, \tag{5.73}$$

we get from equation (5.72) the relation for the eigenvalues

$$Q_\emptyset Q_{12} \propto Q_+^+(u)Q_-^-(u) - Q_-^+(u)Q_+^-(u), \tag{5.74}$$

where we have introduced the shorthand notation

$$g^\pm(u) = g\left(u \pm \tfrac{\mathrm{i}}{2}\right), \qquad\qquad g^{[n]}(u) = g\left(u + n\tfrac{\mathrm{i}}{2}\right), \tag{5.75}$$

for any function g of the spectral parameter. This is an example of what is called a QQ-relation and they will play a central role in chapter 7.

We can depict the QQ-relation by taking the Dynkin diagram of $SU(2)$, placing the two Q-functions $Q_\pm$ at the node and extend it with two more nodes for the trivial Q-functions $Q_\emptyset$ and Q_{12}, as in figure 5.2. Although somewhat trivial for the group $SU(2)$, we can view the QQ-relation (5.74) as connecting the Q-functions along this diagram. This concept will be a lot more useful when considering systems with larger symmetry groups.

A second important aspect of equation (5.72), or the QQ-relation (5.74), is that they contain the full set of Bethe equations for the system. Shifting $u \to u + \frac{\mathrm{i}}{2}$ and dividing both sides of the equation by themselves but with the opposite shift $u \to u - \frac{\mathrm{i}}{2}$ yields the Bethe equations whenever $u \in \mathbf{u}, \mathbf{v}$.

We have thus seen, though briefly, that the Q-operators extend the commuting family in the integrable system and that their eigenvalues are the Q-functions carrying the Bethe roots.

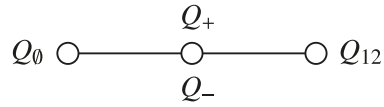


Figure 5.2. The Q-functions of the twisted $SU(2)$ spin-chain placed on an extended Dynkin diagram with the $SU(2)$-node in the middle. They satisfy the QQ-relation of equation (5.74). Diagrams of this type are very useful for keeping track of the QQ-relations for chains with larger symmetry groups, as we will see in chapter 7.

Their actual construction is through a sort of factorization of the generalized Lax operator such that they can be viewed as transfer matrices corresponding to the two Lax operators

$$\mathcal{L}_+(u) = \begin{pmatrix} u-\mathbf{h} & \mathbf{a}^+ \\ -\mathbf{a}^- & 1 \end{pmatrix}, \tag{5.76a}$$

$$\mathcal{L}_-(u) = \begin{pmatrix} 1 & \mathbf{a}^+ \\ \mathbf{a}^- & u-\mathbf{h} \end{pmatrix}, \tag{5.76b}$$

where $\{\mathbf{a}^-, \mathbf{h}, \mathbf{a}^+\}$ make up the harmonic oscillator algebra in the Holstein-Primakoff representation[4] of $\mathfrak{sl}(2)$. From these building blocks, we have

$$Q_\pm(u) = \frac{\mathrm{e}^{\pm\frac{1}{2}\beta L u}\,\mathrm{tr}_a\left[\mathrm{e}^{-\mathrm{i}\beta L\mathbf{h}}\,\mathcal{L}_\pm(u)\cdots\mathcal{L}_\pm(u)\right]}{\mathrm{tr}_a\left[\mathrm{e}^{-\mathrm{i}\beta L\mathbf{h}}\right]}, \tag{5.79}$$

where there are L Lax operators in the product and the traces are taken over the auxiliary Fock space of the oscillator representation.

Although it is not directly apparent from this presentation, it follows from the construction that

$$f(\beta)\,\mathcal{T}_j^+(u) = Q_+^+(u+\mathrm{i}\,j)\,Q_-^-(u-\mathrm{i}\,j), \tag{5.80}$$

and that

$$[\mathcal{T}_j(u), Q_\pm(v)] = [Q_+(u), Q_-(v)] = [Q_\pm(u), Q_\pm(v)] = 0. \tag{5.81}$$

As such, they can be viewed as a sort of factorization of the transfer matrix and, in a sense, the most fundamental objects in the integrable structure of operators.

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[4]The oscillator algebra is as usual $[\mathbf{h}, \mathbf{a}^\pm] = \pm\mathbf{a}^\pm$, $[\mathbf{a}^-, \mathbf{a}^+] = 1$ and $\mathbf{h} = \mathbf{a}^+\mathbf{a}^- + \frac{1}{2}$. The $\mathfrak{sl}(2)$-generators in the j-representation are constructed as

$$J_j^- = \mathbf{a}^+, \qquad J_j^+ = \left(2j - \mathbf{a}^+\mathbf{a}^-\right)\mathbf{a}^-, \qquad J_j^3 = j - \mathbf{a}^+\mathbf{a}^-, \tag{5.77}$$

or

$$J_j^- = \mathbf{a}^+\left(2j - \mathbf{a}^+\mathbf{a}^-\right), \qquad J_j^+ = \mathbf{a}^-, \qquad J_j^3 = j - \mathbf{a}^+\mathbf{a}^-. \tag{5.78}$$

5.7 Higher rank spin-chains

Having reviewed the most important concepts for the $SU(2)$-spin-chain, we can change each local physical Hilbert space $\mathcal{H}_l$ to a representation of $GL(N|K)$. Still, there is a way to extend the success of the Bethe ansatz through what is called the nested Bethe ansatz. This gives equations for $N+K-1$ sets of Bethe roots, $u_k^{(a)}$ with $a = 1, \cdots, N+K-1$, belonging to each node in the Dynkin diagram. Let us exemplify this for $\mathfrak{u}(N|K)$.

The Bethe equations can be summarized very neatly with only two cases, depending on whether the corresponding node is bosonic or fermionic, or in the p-grading, whether $p_a = p_{a+1}$ or not. For the bosonic nodes, the Bethe equations read

$$\prod_{j=1}^{M_a} \frac{u_k^{(a)} - u_j^{(a)} + \mathrm{i}}{u_k^{(a)} - u_j^{(a)} - \mathrm{i}} \prod_{j=1}^{M_{a-1}} \frac{u_k^{(a)} - u_j^{(a-1)} - \frac{\mathrm{i}}{2}}{u_k^{(a)} - u_j^{(a-1)} + \frac{\mathrm{i}}{2}} \prod_{j=1}^{M_{a+1}} \frac{u_k^{(a)} - u_j^{(a+1)} - \frac{\mathrm{i}}{2}}{u_k^{(a)} - u_j^{(a+1)} + \frac{\mathrm{i}}{2}} = -1 \tag{5.82}$$

while for the fermionic nodes, with $p_a \neq p_{a+1}$, they are

$$\prod_{j=1}^{M_{a-1}} \frac{u_k^{(a)} - u_j^{(a-1)} - \frac{\mathrm{i}}{2}}{u_k^{(a)} - u_j^{(a-1)} + \frac{\mathrm{i}}{2}} \prod_{j=1}^{M_{a+1}} \frac{u_k^{(a)} - u_j^{(a+1)} - \frac{\mathrm{i}}{2}}{u_k^{(a)} - u_j^{(a+1)} + \frac{\mathrm{i}}{2}} = 1\,. \tag{5.83}$$

The number of roots M_a of each type naturally depends on what representation the relevant state is part of. Furthermore, the "endpoints" at $a = 0$ and $a = N+K$ are supplemented with fixed trivial roots $u_k^{(0)}$ and $u_k^{(N+K)}$.

The equations (5.22) for $SU(2)$ clearly fits into the formulation, with $M_0 = L$ fixed roots which all are $u_k^{(0)} = 0$ and, at the other end, $M_2 = 0$. Note also how this fits with the trivial Q-functions and the "extended" Dynkin diagram introduced in equations (5.73) and figure 5.2, respectively.

The non-compact case is a little bit different and, as we saw from the 1-loop dilatation operator (2.67), the relevant spin-chain Hamiltonian of $\mathcal{N} = 4$ SYM has $PSU(2,2|4)$ as symmetry group. We will, however, approach the higher rank spin-chain of $PSU(2,2|4)$ from the viewpoint of Q-systems in chapter 7 and will not need the explicit expressions. For further reading on the nested Bethe ansatz and supersymmetric spin-chains, see e.g. [71–73].

5.7.1 Q-operators for the $PSU(2,2|4)$ spin-chain

To construct Q-operators for the full spin-chain corresponding to the singleton representation of $PSU(2,2|4)$ appearing in planar $\mathcal{N} = 4$ SYM, we need to modify the Lax operators (5.76) again. To accommodate all possible excitations, the oscillator formalism for $\mathfrak{sl}(2)$ needs to be promoted to the full set of oscillators of section 2.3.2.

For the actual construction, we refer to [61, 74–77]. There is an additional complication in obtaining the explicit matrix elements as compared to

the $SU(2)$-case. Again, non-compact directions are introduced with the full $PSU(2,2|4)$ and extra care is needed [78].

The larger Dynkin diagram now opens up for a much larger set of QQ-relations. In order to keep track of them, we label each Q-operator with two multi-indices according to the p-grading, A for indices with $p = 0$ and I for indices with $p = 1$. Each Q-operator can then be written as $Q_{A|I}$, where A and I have zero up to four ordered indices ranging from 1 to 4.

We will return to this notation and the QQ-relations in greater detail in chapter 7.

6. $\mathcal{N} = 4$ SYM with a defect

We have already discussed two types of deformations of $\mathcal{N} = 4$ SYM in chapter 3. We now turn our attention to a third, and slightly different, way of modifying this theory while still making use of integrability. We will introduce a defect which in effect will give a setup very reminiscent of CFTs with boundaries, pioneered by Cardy [79].

Such a defect allows for new two-point functions and non-trivial one-point functions which have been the focus of a series of papers exploiting integrability techniques to arrive at compact determinant formulas in terms of Bethe rapidities [9–11, 80–86]. We will review the necessary background from these papers which underlies much of Paper I and II.

6.1 $\mathcal{N} = 4$ SYM with a defect

As we saw in section 2.5, $\mathcal{N} = 4$ SYM is dual to a D3-brane setup in string theory on $AdS_5 \times S^5$. There is a way of introducing a probe[1] D5-brane into the picture with a three-dimensional intersection with the stack of D3-branes, such that a number k of the D3-branes dissolve into the D5-brane. This is due to a background gauge field having k units of flux through an S^2-part of the D5-brane. This brane setup was studied in [7, 87] but our focus will be on the dual field theory [88].

The gauge/gravity duality is still applicable but the inserted D5-brane corresponds to the introduction of a defect into the $\mathcal{N} = 4$ SYM theory. It resides at the orthogonal coordinate value $x_\perp = 0$ and is thus a codimensional-one object, reflecting the intersection between the D3- and the D5-branes. The defect alters the action into the sum

$$S = S_{\mathcal{N}=4} + S_{\text{def}}, \tag{6.1}$$

where the original action of $\mathcal{N} = 4$ still goverens the bulk fields, i.e. fields in the region $x_\perp > 0$, while the second term S_{def} contains new interacting fields confined to the defect and couplings between them and the original fields of $\mathcal{N} = 4$. A sketch of the setup can be found in figure 6.1.

In the present work, we will only stay at treelevel at which the fields of the defect never come into play; hence we leave S_{def} aside [11]. In fact, only the scalar fields from $\mathcal{N} = 4$ SYM will play a role in the following.

[1] meaning that any backreaction is neglected

6.1.1 Symmetries

The introduction of a defect breaks several of the symmetries present in $\mathcal{N} = 4$ SYM. Most obviously, translation symmetry in the $x_\perp$-coordinate is broken and so are the rotations and boosts in the Lorentz group that involve the $x_\perp$-direction. This breaks the Lorentz symmetry from $SO(1,3)$ down to $SO(1,2) \simeq SU(2)$. Similarily, only the conformal transformations that map $x_\perp = 0$ onto itself survive, that is dilatations and three of the special conformal transformations. The four-dimensional conformal group $SO(4,2)$ is thus reduced to $SO(3,2)$.

As the superchargers anticommute into translations, there is also a loss of supersymmetry. Only half of the supercharges are preserved and the same goes for the superconformal charges. Consequently, the R-symmetry group is reduced from $SU(4)$ to $SO(3)\times SO(3)$ and the maximally remaining symmetry of the total group $PSU(2,2|4)$ is $OSP(4|4)$. Depending on the explicit action on the defect, there may be even more symmetries broken when introducing a defect.

6.1.2 Correlation functions

The broken conformal symmetries relax some of the restrictions put on the correlation functions, discussed in section 2.4. As translation along the $x_\perp$-coordinate no longer is a symmetry, the defect CFT now allows for one-point functions of the form

$$\langle \mathcal{O} \rangle = \frac{C}{x_\perp^{\Delta_0}}, \tag{6.2}$$

for some constant C, which is a meaningful quantity if we still demand canonical normalization of the two-point functions, as in equation (2.54).

The orthogonal coordinate also provides an additional conformal cross-ratio

$$\xi = \frac{|x-y|^2}{4x_\perp y_\perp} \tag{6.3}$$

and permits non-vanishing two-point functions between operators of different scaling dimensions

$$\langle \mathcal{O}_A(x)\mathcal{O}_B(y)\rangle = \frac{f(\xi)}{x_\perp^{\Delta_A} y_\perp^{\Delta_B}}, \tag{6.4}$$

with f being some function of the cross-ratio.

Note that both these go to zero when the distance from the defect is comparably large, in accordance with the correlation functions in a theory without a defect or a boundary.

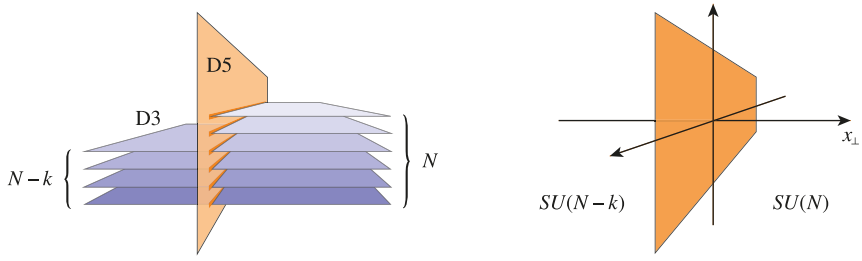


Figure 6.1. The left illustrates the D-brane construction in the holographic dual, where k D3-branes are absorbed into the D5-brane and the three-dimensional intersection corresponds to the defect in $\mathcal{N}=4$ SYM. The latter is showed to the right as a codimensional-one surface (here as a projection onto three of the four dimensions) residing at $x_\perp = 0$ separating the two regions of different gauge groups $SU(N-k)$ for $x_\perp < 0$ and (broken) $SU(N)$ for $x_\perp > 0$.

6.1.3 Vacuum solution

The defect also breaks the gauge symmetry in that it is compatible with non-zero vacuum expectation values for the scalars. On the $x_\perp < 0$ side of the defect, the gauge group is broken from $SU(N)$ down to $SU(N-k)$ by infinite scalar expectation values. This is where the number k of absorbed D3-branes in the string theory dual appears in the gauge theory. It can be any non-negative integer and constitutes thus a new scalable parameter which is not there for the original theory.

On the other side where $x_\perp > 0$, the gauge group is a spontaneously broken $SU(N)$ where again only $SU(N-k)$ effectively remains. The field theory situation and its correspondence with the brane setup are, as stated, illustrated in figure 6.1.

For the explicit classical solution, we turn to the equations of motion which for the complex scalars in $S_{\mathcal{N}=4}$ read

$$\frac{\partial^2}{\partial x_\perp^2}\phi^i = \Big[\phi^j, \Big[\phi^j, \phi^i\Big]\Big]. \tag{6.5}$$

They are implied by the stricter Nahm equation

$$\frac{\partial}{\partial x_\perp}\phi^i = \frac{\mathrm{i}}{2}\varepsilon^{ijk}\Big[\phi^j, \phi^k\Big]. \tag{6.6}$$

It has a classical solution in which three of the six *real* scalars φ_i acquire a non-zero expectation value

$$\varphi_i^{\mathrm{cl}} = -\frac{1}{x_\perp}\, t_i \oplus \mathbb{0}_{(N-k)}, \quad i = 1,2,3, \tag{6.7}$$

while the other φ_j^{cl} with $j = 4,5,6$, are zero. Here, we have introduced the shorthand $\mathbb{0}_{(n)}$ for the $(n\times n)$-dimensional zero matrix and the generators t_i of

$SU(2)$ in a unitary $(k\times k)$-dimensional representation. As such, they satisfy the well-known algebra

$$[t_i, t_j] = \mathrm{i}\varepsilon_{ijk}t_k. \tag{6.8}$$

This split of the six scalars reflects the R-symmetry now being $SO(3)\times SO(3)$.

Representation for t_i

The $(k\times k)$-dimensional representation for t_i can be constructed from the single-entry matrices $E^i{}_j$, which are zero everywhere except of a 1 at position (i, j). Dressing them with the constants

$$c_i = \sqrt{i(k-i)}, \qquad d_i = \frac{1}{2}(k-2i+1), \tag{6.9}$$

we can build the matrices

$$t_+ = \sum_{i=1}^{k-1} c_i E^i{}_{i+1}, \qquad t_- = \sum_{i=1}^{k-1} c_i E^{i+1}{}_i. \tag{6.10}$$

These are the common raising and lowering matrices such that the standard t_i in equation (6.8) are

$$t_1 = \frac{t_+ + t_-}{2}, \qquad t_2 = \frac{t_+ - t_-}{2\mathrm{i}}, \qquad t_3 = \sum_{i=1}^{k} d_i E^i{}_i. \tag{6.11}$$

These matrices satisfy the transposition relations

$$t_1^{\mathrm{T}} = t_1, \qquad t_2^{\mathrm{T}} = -t_2, \qquad t_3^{\mathrm{T}} = t_3, \tag{6.12}$$

which will be used later when studying the integrable properties of this theory.

When $k = 2$.

In the special case of $k = 2$, the t_i are just the ordinary Pauli matrices $t_i = \frac{\sigma_i}{2}$ and satisfy the Clifford algebra in three dimensions. On top of the transposition relations above, they hence have the special products

$$t_1^2 = t_2^2 = t_3^2 = \frac{1}{4}, \qquad \{t_i, t_j\} = 0 \quad \text{for } i \neq j, \qquad (k = 2). \tag{6.13}$$

6.2 Treelevel one-point functions

The classical solution in the defect theory allows composite single-trace operators built only out of scalars to have non-zero treelevel one-point functions. Thanks to integrability surviving in the defect theory, there is a neat determinant formula for these expressions. It is valid for the full scalar sector but let us focus on the $SU(2)$-subsector and build it step by step. This restriction is also what is relevant for Paper I and II.

6.2.1 $SU(2)$-subsector

As mentioned in section 2.1, the $SU(2)$-subsector consists of operators built out of two complex scalars, which inside the trace are interpreted as sites with spin-up and spin-down, respectively. Let us choose

$$\phi_1 \leftrightarrow |\uparrow\rangle \qquad \text{and} \qquad \phi_2 \leftrightarrow |\downarrow\rangle\,. \tag{6.14}$$

An operator in the $SU(2)$-subsector of length L can then be written as

$$\mathcal{O}_\Psi = \Psi^{i_1\dots i_L}\,\mathrm{tr}\left(\phi_{i_1}\phi_{i_2}\dots\phi_{i_L}\right), \tag{6.15}$$

where $i_\ell = 1,2$ and the Bethe wavefunction Ψ from the ansatz (5.11) ensures a well-defined scaling dimension.

Substituting the classical solution for $\phi^{\mathrm{cl}}_{i_\ell}$, we obtain the treelevel one-point function

$$\langle\mathcal{O}_\Psi\rangle_{\mathrm{tree}} = \frac{(-1)^L}{x_\perp^L}\Psi^{i_1\dots i_L}\,\mathrm{tr}\left(t_{i_1}\dots t_{i_L}\right) = (-1)^L\left(\frac{1}{2g}\right)^{\frac{L}{2}}\frac{1}{\sqrt{L}}\,\frac{C_\Psi}{x_\perp^{\Delta_0}}\,, \tag{6.16}$$

in accordance with the general form (6.2).

We can now reinterpret this expression as a scalar product in the spin-chain picture, which will pave the way to the determinant formula. The normalization of the constant C_Ψ was chosen with this in mind.

6.2.2 One-point functions as spin-chain overlaps with an MPS

Looking at the one-point function, we device a spin-chain state that through a scalar product with a Bethe state precisely reproduces C_Ψ in equation (6.16). Evidently, that state has to be

$$\langle\mathrm{MPS}| = \mathrm{tr}\left(\langle\uparrow|t_1 + \langle\downarrow|t_2\right)^{\otimes L}, \tag{6.17}$$

where the trace is over the resulting product of t_is. Recalling the discussion about matrix product states in section 5.4, this clearly falls into the category.

The constant C_Ψ in the one-point function (6.16) can now be written as

$$C_\Psi = \frac{\langle\mathrm{MPS}|\Psi\rangle}{\sqrt{\langle\Psi|\Psi\rangle}}\,, \tag{6.18}$$

where the Bethe state $|\Psi\rangle$ obviously corresponds to the single-trace operator in (6.16).

The overlap (6.18) is normalized but is still sensitive to the overall phase of the Bethe state. We remove this ambiguity simply by always requiring C_Ψ to be real and positive.

Selection rules

From equation (6.18), we immediately get some selection rules for overlaps with Bethe states.

- Both the length L and the number of excitations M in $|\Psi\rangle$ need to be even. This follows from the traces of t_is and the fact there are two similarity transformations[2] U and V of t_i with the properties

$$Ut_1U^{-1} = t_1\,, \quad Ut_{2,3}U^{-1} = -t_{2,3}\,, \quad Vt_{1,2} = -t_{1,2}\,, \quad Vt_3V^{-1} = t_3\,. \tag{6.20}$$

  Inserting these into the traces instantly yields $\mathrm{tr}(\dots) = (-1)^M \mathrm{tr}(\dots)$ and $\mathrm{tr}(\dots) = (-1)^L \mathrm{tr}(\dots)$.

- The MPS is integrable in the sense of equation (5.48). As such, it is annihilated by all odd charges Q_{2n+1} and only unpaired[3] Bethe states can have a non-zero overlap, i.e. the rapidities necessarily have the structure

$$\mathbf{u} = \{u_1, u_2, \dots, u_{M/2}, -u_1, \dots, -u_{M/2}\}\,. \tag{6.21}$$

Overlaps with descendants

Recall the discussion about (Bethe) descendants in section 5.5.2. The overlap with the MPS of such a state is expressible in terms of the overlap with its highest weight state. Explicit calculations suggest that

$$\langle \mathrm{MPS}|(S^-)^n|\Psi_M\rangle = \frac{n!(\frac{L}{2}-M)!}{\left(\frac{n}{2}\right)!\left(\frac{L-2M-n}{2}\right)!}\langle \mathrm{MPS}|\Psi_M\rangle\,, \tag{6.22}$$

as the action on the MPS in such an overlap is

$$(S^+)^n|\mathrm{MPS}\rangle = \frac{n!(\frac{L}{2}-M)!}{\left(\frac{n}{2}\right)!\left(\frac{L-2M-n}{2}\right)!}|\mathrm{MPS}\rangle + S^-|\dots\rangle\,. \tag{6.23}$$

The second term is orthogonal to the highest weight state leaving only the first in equation (6.22). [85]

6.2.3 Overlap formula

We will work our way up to the final determinant formula for the $SU(2)$-subsector step by step, following the actual process in which it was found.

[2]The explicit matrices are

$$U = U^{-1} = \sum_{i=1}^{k} E^i{}_{k-i+1}\,, \qquad V = V^{-1} = \sum_{i=1}^{k} (-1)^i E^i{}_i\,. \tag{6.19}$$

[3]As the spin-chain Hamiltonian is parity invariant, all energy states come in parity pairs except parity invariant eigenstates. They are hence called unpaired all though the invariance ironically implies a pair structure for the momenta.

First we will look at the spin-chain vacuum, then the special case $k = 2$, from which we will build a recursion formula to reach the full result in (6.37).

The vacuum

The simplest state to consider is the spin-chain vacuum $|\Omega\rangle$ which corresponds to the BPS-operator $\mathrm{tr}(Z^L)$. The overlap with the MPS is

$$C_{\Omega,k} = \frac{\langle \mathrm{MPS}_k|\Omega\rangle}{\sqrt{\langle\Omega|\Omega\rangle}} = \mathrm{tr}\left(t_1^L\right) \overset{(6.11)}{=} \sum_{i=1}^{k} d_i^L \,. \tag{6.24}$$

This can be written in the form of a Bernoulli polynomial with index $L+1$,

$$C_{\Omega,k} = -\frac{2}{L+1} B_{L+1}\left(\tfrac{1-k}{2}\right), \tag{6.25}$$

from which it is clear that $C_{\Omega,k}$ is a polynomial in k of degree $L+1$.

The case $k = 2$ and general M

At the heart of the determinant formula lies a relationship between the overlap with the MPS and the norm of the Bethe state. Recall the Gaudin norm (5.26) of a coordinate Bethe state. For an unpaired state with rapidities on the form (6.21), the matrix simplifies to the block form

$$\frac{\partial \Phi_i}{\partial u_j} = \begin{pmatrix} G_1 & G_2 \\ G_2 & G_1 \end{pmatrix}, \tag{6.26}$$

such that the determinant factorizes into

$$\det\left(\partial_{u_j}\Phi_i\right) = \det G^+ \det G^- \,. \tag{6.27}$$

Explicitly,

$$\begin{aligned} G^{\pm}_{ij} = \left[G_1 \pm G_2\right]_{ij} &= \frac{2}{(u_i-u_j)^2+1} \pm \frac{2}{(u_i+u_j)^2+1} \\ &+ \left[\frac{L}{u_i^2+1/4} - \sum_{l=1}^{M/2}\left(\frac{2}{(u_i-u_l)^2+1} + \frac{2}{(u_i+u_l)^2+1}\right)\right]\delta_{ij} \,. \end{aligned} \tag{6.28}$$

The remarkable relation is that the overlap with the MPS can be written as

$$\langle \mathrm{MPS}|\Psi\rangle = 2^{1-L}\sqrt{\prod_{i=1}^{M/2}\frac{u_i^2+1/4}{u_i^2}}\prod_{i=1}^{M/2}\left(u_i^2+1/4\right)\det G^+ \,. \tag{6.29}$$

Dividing with the norm gives the formula for $k = 2$

$$C_{k=2} = 2^{1-L}\sqrt{\prod_{i=1}^{M/2}\frac{u_i^2+1/4}{u_i^2}\frac{\det G^+}{\det G^-}} = 2^{1-L}\sqrt{\frac{Q(\frac{\mathrm{i}}{2})}{Q(0)}\frac{\det G^+}{\det G^-}}, \tag{6.30}$$

repackaged in the Q-functions of section 5.6.

General k

The result (6.30) for $k = 2$ can be extended to general k by a recursion relation. To arrive at it, we need the transfer matrix of the spin-chain, but built from a slightly changed Lax operator,

$$\mathcal{T}(u) = \mathrm{tr}_a(\mathcal{L}_{aL}\cdots\mathcal{L}_{a1}), \qquad \mathcal{L}(u) = \mathcal{I} + \frac{\mathrm{i}}{u-\frac{\mathrm{i}}{2}}\mathcal{P}. \tag{6.31}$$

The action of the Lax operator on an auxiliary space and $\mathcal{H}_\ell$ is

$$\mathcal{L}_{a\ell}\left(\tfrac{\mathrm{i}k}{2}\right)\left(t_1^{(k)}|\uparrow_\ell\rangle + t_2^{(k)}|\downarrow_\ell\rangle\right) = \tau_1^{(k)}|\uparrow_\ell\rangle + \tau_2^{(k)}|\downarrow_\ell\rangle \tag{6.32}$$

with the $(2k\times 2k)$-dimensional matrices

$$\tau_1^{(k)} = \begin{pmatrix} \frac{k+1}{k-1}t_1^{(k)} & 0 \\ \frac{2}{k-1}t_2^{(k)} & t_1^{(k)} \end{pmatrix}, \qquad \tau_2^{(k)} = \begin{pmatrix} t_2^{(k)} & \frac{2}{k-1}t_1^{(k)} \\ 0 & \frac{k+1}{k-1}t_2^{(k)} \end{pmatrix}. \tag{6.33}$$

We can shift the ks within the $\tau_i^{(k)}$ by means of a similarity transformation A, to be found in [81]. The effect is

$$A\tau_i^{(k)}A^{-1} = \begin{pmatrix} t_i^{(k+2)} & 0 \\ *_i & \frac{k+1}{k-1}t_i^{(k-2)} \end{pmatrix}, \tag{6.34}$$

where we denote the irrelevant lower left block with $*$ in blissful ignorance, granted by the trace and the upper right vanishing block.

The transfer matrix trace over the product of such matrices then provides a recursion relation for the MPS,

$$|\mathrm{MPS}_{k+2}\rangle = \mathcal{T}\left(\tfrac{\mathrm{i}k}{2}\right)|\mathrm{MPS}_k\rangle - \left(\frac{k+1}{k-1}\right)^L|\mathrm{MPS}_{k-2}\rangle. \tag{6.35}$$

Inserting this into the overlap together with the transfer matrix eigenvalue (5.42) leads to the sought after relation for the constant C_k

$$C_{k+2} = \Lambda\left(\tfrac{\mathrm{i}k}{2}\,|\,\Psi\right)C_k - \left(\frac{k+1}{k-1}\right)^L C_{k-2}. \tag{6.36}$$

The solution to this relation gives the final expression for the determinant formula in the $SU(2)$-subsector

$$C_k = \mathrm{i}^L\mathcal{T}_{k-1}(0)\sqrt{\frac{Q(\frac{\mathrm{i}}{2})Q(0)}{Q^2(\frac{\mathrm{i}k}{2})}\,\frac{\det G^+}{\det G^-}}, \tag{6.37}$$

where

$$\mathcal{T}_n(u) = \sum_{l=-\frac{n}{2}}^{\frac{n}{2}}(u+\mathrm{i}l)^L\frac{Q\left(u+\frac{n+1}{2}\mathrm{i}\right)Q\left(u-\frac{n+1}{2}\mathrm{i}\right)}{Q\left(u+(l-\frac{1}{2})\mathrm{i}\right)Q\left(u+(l+\frac{1}{2})\mathrm{i}\right)}. \tag{6.38}$$

The recursion relation only proves this formula for even k but it holds also for all odd k that has been explicitly calculated.

C_3 and C_2 are related through Q-functions,

$$C_3 = 2^L \frac{Q(0)}{Q(\frac{\mathrm{i}}{2})} C_2 \,, \tag{6.39}$$

but a proof of formula (6.37) for odd k is yet missing.

Large k

We noted above that the vacuum overlap (6.25) is a polynomial in k of degree $L+1$. The determinant formula (6.37) allows us to note the scaling with k for any number of excitations.

If both L and M are of order 1 but we let k be large, we can approximate the sum in C_k as

$$C_k \overset{k\to\infty}{\sim} 2^{L-M} C_2 \prod_{i=1}^{M/2} u_i^2 \sum_{l=1}^{k/2} l^{L-2M} \,, \tag{6.40}$$

which gives us the general overlap scaling

$$C_k \overset{k\to\infty}{\sim} \left(\frac{2^{M-1}}{L-2M+1} C_2 \prod_{i=1}^{M/2} u_i^2 \right) k^{L-M+1} + O(k^{L-M}) \,. \tag{6.41}$$

This power law will no longer hold below when we consider the β-deformed version of this setup.

6.2.4 Larger subsectors

The determinant formula (6.37) has consecutively been extend. The first result for the $SU(3)$-subsector was found in [82] while the full $SO(6)$-scalar sector was covered by [84]. It has thus been shown that the MPS which captures the defect is integrable for the entire scalar sector and that any treelevel one-point function of single-trace operators built from scalars can be written in a determinant form depending on the rapidities. As the $SU(2)$-result is sufficient for our purposes, we refer to the references for the actual formulas.

6.3 Propagators in presence of the defect

Having discussed the treelevel one-point functions, the next natural step is to consider two-point functions. That necessarily involve knowledge of the propagators which, in presence of the defect, turns out to be a non-trivial task. They were worked out in detail in [11] from which we repeat the, for us, most relevant results.

To study the propagating fields, we split the scalars into their classical and propagating parts,

$$\varphi_i = \varphi_i^{\mathrm{cl}} + \tilde{\varphi}_i \,. \tag{6.42}$$

The action consequently splits into kinetic terms and "mass" terms, plus a number of new cubic and quartic interactions. Restricting to terms only containing scalars, we get

$$S_{\varphi,\mathrm{kin}} = \frac{2}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4 x \ \mathrm{tr}\left(-\frac{1}{2} \partial_\mu \tilde{\varphi}_i \partial^\mu \tilde{\varphi}_i \right), \tag{6.43}$$

$$\begin{aligned} S_{\varphi,\mathrm{mass}} = \frac{2}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4 x \ \frac{1}{2} \mathrm{tr}\Big(& [\varphi_i^{\mathrm{cl}}, \varphi_j^{\mathrm{cl}}][\tilde{\varphi}_i, \tilde{\varphi}_j] + [\varphi_i^{\mathrm{cl}}, \tilde{\varphi}_j][\varphi_i^{\mathrm{cl}}, \tilde{\varphi}_j] \\ & + [\varphi_i^{\mathrm{cl}}, \tilde{\varphi}_j][\tilde{\varphi}_i, \varphi_j^{\mathrm{cl}}] + [\varphi_i^{\mathrm{cl}}, \tilde{\varphi}_i][\varphi_j^{\mathrm{cl}}, \tilde{\varphi}_j] \Big). \end{aligned} \tag{6.44}$$

Already looking at the scalars alone, we see that the classical values, interpreted as masses, generate a lot of mixing. The mass matrix is not diagonal in flavor nor color indices and in the full action there is also a mass mixing between the scalars and the gauge field, due to the covariant derivative.

We use the word "mass" in a wide sense here as the classical values have a coordinate dependence in form of $x_\perp^{-1}$. We shall nevertheless see that these terms produce actual masses in a reinterpretation of the propagators as being inside an AdS geometry.

6.3.1 Diagonalizing the field type and flavor indices

We can see the structure of the mass mixing in a clearer way if we split the scalars into the two $SO(3)$-representations $\tilde{\varphi}_i$, with $i = 1,2,3$, and those with capitalized indices $\tilde{\varphi}_I$, for which $I = 4,5,6$. We replace the classical values with the expression (6.7) and introduce the notation[4]

$$L_i = \mathrm{ad}(t_i) = -x_\perp [\varphi_i^{\mathrm{cl}}, \cdot] \,. \tag{6.45}$$

The cyclicity of the traces lets us rearrange the commutators in the action (6.44) into

$$\mathrm{tr}\Big(-\tilde{\varphi}_I L^2 \tilde{\varphi}_I - \tilde{\varphi}_i L^2 \tilde{\varphi}_i + 2\mathrm{i}\varepsilon_{ijk} \tilde{\varphi}_i L_j \tilde{\varphi}_k \Big). \tag{6.46}$$

Writing this with the flavor space matrices

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.47}$$

[4] We will in the following let it be understood whether the $(N-k)\times(N-k)$ block of zeros is included or not.

which satisfy the $SU(2)$-algebra

$$[S_i, S_j] = \mathrm{i}\varepsilon_{ijk} S_k\,, \tag{6.48}$$

it is now clear that the mass mixing of the scalars has the same form as the text book diagonalization of electrons in an atom. In this analogy, the scalars $\tilde{\varphi}_I$ only have an orbital angular momentum while $\tilde{\varphi}_i$ experience a spin-orbit coupling:

$$\mathrm{tr}\Big(-\tilde{\varphi}_I L^2 \tilde{\varphi}_I + \Phi^{\mathrm{T}}(-L^2 + 2L\cdot S)\Phi\Big). \tag{6.49}$$

Here we temporarily wrote the vector $\Phi = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)^{\mathrm{T}}$.

This structure persists when diagonalizing the full (bulk) action of the theory where also the gauge field enters the bosonic part. We pack the bosonic fields into the two collecting vectors

$$\Phi_{\mathrm{orb}} = (A_0, A_1, A_2, \tilde{\varphi}_4, \tilde{\varphi}_5, \tilde{\varphi}_6)^{\mathrm{T}}, \qquad \Phi_{\text{s-o}} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, A_3)^{\mathrm{T}}, \tag{6.50}$$

which lets the full bosonic mass part of the action be written as

$$S_{\mathrm{b,mass}} = \frac{2}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\, \frac{1}{x_\perp^2}\,\mathrm{tr}\Big(\Phi_{\mathrm{orb}}^{\mathrm{T}}(-\frac{1}{2}L^2)\Phi_{\mathrm{orb}} + \Phi_{\text{s-o}}^{\mathrm{T}}\Big(-\frac{1}{2}L^2 + 2L\cdot S\Big)\Phi_{\text{s-o}}\Big) \tag{6.51}$$

where now the "spin"-matrices need to be updated to

$$S_1 = -\frac{1}{2}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad S_2 = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \qquad S_3 = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{6.52}$$

They again satisfy the $SU(2)$-algebra and their irreducible decomposition[5] brings $\Phi_{\text{s-o}}$ into two spin-1/2 representations

$$U^\dagger S_i U = \begin{pmatrix} \frac{1}{2}\sigma_i & 0 \\ 0 & \frac{1}{2}\sigma_i \end{pmatrix}, \qquad \Phi_{\text{s/o}} := U^\dagger \Phi_{\text{s-o}} = \begin{pmatrix} \mathrm{i}\tilde{\varphi}_1 + \tilde{\varphi}_2 \\ -\mathrm{i}\tilde{\varphi}_3 - A_3 \\ -\mathrm{i}\tilde{\varphi}_3 + A_3 \\ -\mathrm{i}\tilde{\varphi}_1 + \tilde{\varphi}_2 \end{pmatrix}. \tag{6.54}$$

This prepares the field types and flavor indices such that the propagators can be diagonalized through the well-known procedure of addition of angular momenta. But for this we also need an appropriate basis in color space.

[5] The basis change is done through

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{i} & 0 & 0 & \mathrm{i} \\ 1 & 0 & 0 & 1 \\ 0 & \mathrm{i} & \mathrm{i} & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}. \tag{6.53}$$

6.3.2 A matrix basis for L_i and fuzzy spherical harmonics

Viewed from the lens of non-commutative geometry, the operator L^2 is the Laplacian on the fuzzy sphere. It is diagonalized by the square matrices called the fuzzy spherical harmonics $\hat{Y}_\ell^m$ which have the orbital and magnetic eigenvalues ℓ and m, respectively. Much in accordance with the regular spherical harmonics, we have

$$L^2 \hat{Y}_\ell^m = \ell(\ell+1)\hat{Y}_\ell^m, \qquad L_3 \hat{Y}_\ell^m = m\hat{Y}_\ell^m. \tag{6.55}$$

If we decompose the fields in color space into the upper left $k\times k$ block with indices s, s', the lower right $(N-k)\times(N-k)$ block with indices a, a' and the remaining two rectangular parts (also isolating the trace)

$$\begin{aligned}\phi = {}&[\phi]^s{}_{s'} E^s{}_{s'} + [\phi]^a{}_{a'} E^a{}_{a'} + \phi_{\mathrm{tr}}((N-k)\mathbb{1}_k + k\mathbb{1}_{N-k}) \\ &+ [\phi]^s{}_a E^s{}_a + [\phi]^a{}_s E^a{}_s,\end{aligned} \tag{6.56}$$

we can make use of the fuzzy spherical harmonics in the $k\times k$ block:

$$[\phi]^s{}_{s'} = \sum_{\ell=1}^{k-1} \sum_{m=-\ell}^{\ell} \phi_{\ell,m} [\hat{Y}_\ell^m]^s{}_{s'}. \tag{6.57}$$

Here ϕ represents any of the field combinations resulting from the flavor diagonalization in the previous section.

The $(N-k)\times(N-k)$ block and the trace part are both annihilated by L_i and are hence massless. The rectangular parts transform in the k-dimensional representation of $SU(2)$ but we will actually not need them for this thesis. As we only will consider treelevel two-point functions, there will only be a single Wick contraction in each operator such that the product of the remaining classical scalar fields will kill everything but the upper $k\times k$ block (see for example equation (6.76) below).

We will however need some more properties of $\hat{Y}_\ell^m$.

Some identities for the fuzzy spherical harmonics

The fuzzy spherical harmonics form an orthogonal basis for the $(k\times k)$-dimensional traceless matrices. Their explicit components are

$$[\hat{Y}_\ell^m]^s{}_{s'} = (-1)^{k-s}\sqrt{2\ell+1}\begin{pmatrix} \frac{k-1}{2} & \ell & \frac{k-1}{2} \\ s-\frac{k+1}{2} & m & -s'+\frac{k+1}{2}\end{pmatrix}, \tag{6.58}$$

expressed in terms of the Wigner $3j$ symbol.

Their orthogonality implies

$$\mathrm{tr}\left[(\hat{Y}_\ell^m)^\dagger \hat{Y}_{\ell'}^{m'}\right] = \delta_{\ell\ell'}\delta_{mm'} \qquad \text{or} \qquad \mathrm{tr}\left[\hat{Y}_\ell^m \hat{Y}_{\ell'}^{m'}\right] = (-1)^m \delta_{\ell\ell'}\delta_{m+m',0}, \tag{6.59}$$

since $\left(\hat{Y}_\ell^m\right)^\dagger = (-1)^m \hat{Y}_\ell^{-m}$. They transform in the spin-ℓ representation of $SU(2)$ and as tensor operators, they obey

$$L_i \hat{Y}_\ell^m = [t_i^{(k)}, \hat{Y}_\ell^m] = \sum_{m'=-\ell}^{\ell} \hat{Y}_\ell^{m'} \left[t_i^{(2\ell+1)}\right]^{\ell-m'+1}{}_{\ell-m+1} = m\hat{Y}_\ell^m, \tag{6.60}$$

where $t_i^{(n)}$ naturally is the t_i generator in the n-dimensional representation.

Finally, we write out the explicit linear combinations of fuzzy spherical harmonics that make up the $SU(2)$-generators;

$$t_1 = c_k\left(\hat{Y}_1^{-1} - \hat{Y}_1^1\right), \tag{6.61a}$$

$$t_2 = \mathrm{i}c_k\left(\hat{Y}_1^{-1} + \hat{Y}_1^1\right), \qquad c_k = \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{6}}, \tag{6.61b}$$

$$t_3 = \sqrt{2}c_k\hat{Y}_1^0. \tag{6.61c}$$

6.3.3 The bosonic propagators

Having completed the diagonalizing procedure for the bosons above, each field component ϕ in Φ_{orb} and $\Phi_{\mathrm{s/o}}$ now appears in the Lagrangian as

$$-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\frac{\mathrm{m}^2}{x_\perp^2}\phi^2, \tag{6.62}$$

where the mass m^2 is $\ell(\ell+1)$ for the $k\times k$ block of the fields in Φ_{orb}, $\ell = 1,\ldots,k-1$, and $\ell(\ell-1)$ and $(\ell+1)(\ell+2)$ for the same block of the fields in $\Phi_{\mathrm{s/o}}$.[6]

The scalar propagator corresponding to the expression (6.62) is then the solution to

$$\left(-\partial_\mu\partial^\mu + \frac{\mathrm{m}^2}{x_\perp^2}\right)K^{\mathrm{m}^2}(x,y) = \frac{g_{\mathrm{YM}}^2}{2}\delta(x-y). \tag{6.63}$$

This expression can be related to the propagator in AdS_4, as alluded to above. We will now show how.

Rephrasing the propagator for a general dimension d as

$$K^{\mathrm{m}^2}(x,y) = \frac{g_{\mathrm{YM}}^2}{2}\frac{\tilde{K}(x,y)}{(x_\perp y_\perp)^{\frac{d-1}{2}}}, \tag{6.64}$$

we get an equation for $\tilde{K}$:

$$\left(-x_\perp^2\partial_\mu\partial^\mu + (d-1)x_\perp\partial_\perp + \mathrm{m}^2 - \frac{d^2-1}{4}\right)\tilde{K}(x,y) = x_\perp^{d+1}\delta(x-y). \tag{6.65}$$

[6] See [11] for a derivation of the latter expressions

This is in fact the same as for the propagator in AdS_{d+1} in the Euclidean coordinates

$$g_{\mu\nu} = \frac{1}{x_\perp^2}\delta_{\mu\nu}\,, \qquad\qquad \sqrt{g} = \frac{1}{x_\perp^{d+1}}\,, \tag{6.66}$$

as is seen from

$$\left(-\nabla_\mu\nabla^\mu + \tilde{\mathrm{m}}^2\right) K_{AdS}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} \tag{6.67}$$

$$\Longleftrightarrow$$

$$\left(-x_\perp^2\partial_\mu\partial^\mu + (d-1)x_\perp\partial_\perp + \tilde{\mathrm{m}}^2\right) K_{AdS}(x,y) = x_\perp^{d+1}\delta(x-y)\,.$$

We hence conclude that

$$K^{\mathrm{m}^2}(x,y) = \frac{g_{\mathrm{YM}}^2}{2}\frac{K_{AdS}(x,y)}{(x_\perp y_\perp)^{\frac{d-1}{2}}}\,, \qquad \text{with} \qquad \tilde{\mathrm{m}}^2 = \mathrm{m}^2 - \frac{d^2-1}{4}\,. \tag{6.68}$$

One explicit representation of the AdS-propagator is

$$K_{AdS}(x,y) = (x_\perp y_\perp)^{\frac{d}{2}} \int \frac{\mathrm{d}^d\vec{k}}{p\pi}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y}}I_\nu(kx_\perp^<)\,K_\nu(kx_\perp^>)\,. \tag{6.69}$$

Here (only) we have temporarily set $k = |k|$ and where $x_\perp^<$ and $x_\perp^>$ is the smaller and larger of $x_\perp$ and $y_\perp$, respectively. I_ν and K_ν are the modified Bessel functions with

$$\nu = \sqrt{\mathrm{m}^2 + \tfrac{1}{4}}\,. \tag{6.70}$$

The general d is kept for the usefulness in the dimension regularization but for our purposes we can replace it with $d = 3$.

6.3.4 The scalar propagators in the flavor basis

The final step to obtain the propagators for $\tilde{\varphi}_i$ and $\tilde{\varphi}_I$ is to back-track the diagonalization procedure and write them in linear combinations of the propagators for the fields in $\Phi_{\mathrm{s/o}}$ and Φ_{orb}. We refer again to [11] for the details while we present the final result for the scalars:

$$\begin{aligned}\langle[\tilde{\varphi}_i(x)]^s{}_{s'}[\tilde{\varphi}_j(y)]^r{}_{r'}\rangle &= \delta_{ij}\sum_{\ell,m}\left[\hat{Y}_\ell^m\right]^s{}_{s'}\left[(\hat{Y}_\ell^m)^\dagger\right]^r{}_{r'}K_1^\ell(x,y)\\ &\quad - \mathrm{i}\varepsilon_{ijk}\sum_{\ell,m,m'}\left[\hat{Y}_\ell^m\right]^s{}_{s'}[(\hat{Y}_\ell^{m'})^\dagger]^r{}_{r'}\left[t_k^{(2\ell+1)}\right]^{\ell-m+1}_{\ell-m'+1}K_2^\ell(x,y)\,,\end{aligned} \tag{6.71}$$

where

$$K_1^\ell(x,y) = \frac{\ell+1}{2\ell+1}K^{\ell(\ell-1)}(x,y) + \frac{\ell}{2\ell+1}K^{(\ell+1)(\ell+2)}(x,y)\,, \tag{6.72}$$

$$K_2^\ell(x,y) = \frac{1}{2\ell+1}\left(K^{\ell(\ell-1)}(x,y) - K^{(\ell-1)(\ell+2)}(x,y)\right), \tag{6.73}$$

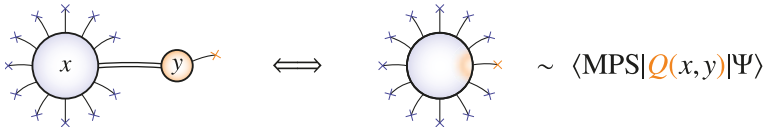


Figure 6.2. A single contraction between a length L $SU(2)$-operator and a length-two operator results in an expression interpretable as a spacetime-dependent operation in the spin-chain corresponding to the one-point function of the longer operator. The xs indicate the classical fields.

and we by $K^{\ell(\ell-1)}$ mean $K^{\mathrm{m}^2=\ell(\ell-1)}$, etc. The other three scalars have the diagonal propagators

$$\langle [\tilde{\varphi}_I(x)]^s{}_{s'} [\tilde{\varphi}_J(y)]^r{}_{r'} \rangle = \delta_{IJ} \sum_{m=-\ell}^{\ell} \left[\hat{Y}_\ell^m\right]^s{}_{s'} \left[(\hat{Y}_\ell^m)^\dagger\right]^r{}_{r'} K^{\ell(\ell+1)}(x,y)\,. \tag{6.74}$$

With the propagators in our hands we can move on to the work and results of Paper I.

6.4 Treelevel two-point functions with length two operator

There exists an interesting special case of two-point functions in the defect $\mathcal{N}=4$ SYM theory that combines all the aspects we have covered in this chapter. Namely, the two-point function between an $SU(2)$-subsector operator $\mathcal{O}_\Psi$, as in equation (6.15), and an operator built out of any two (complex) scalars,

$$\mathcal{O}_{Y_1Y_2} = \mathrm{tr}(Y_1Y_2)\,, \qquad Y_1, Y_2 \in \{\phi_1, \phi_2, \phi_3, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3\}\,. \tag{6.75}$$

Note that we have redefined the symbol Y in this section to denote any complex scalar.

A contraction between these two operators can be interpreted as an operator in the spin-chain picture that corresponds to the one-point function of $\mathcal{O}_\Psi$, conceptually illustrated in figure 6.2. Let us sketch how that comes about; the details and full result are to be found in Paper I.[7]

The one-contraction two-point function reads

$$\begin{aligned}\langle \mathcal{O}_\Psi \mathcal{O}_{Y_1Y_2} \rangle_{1\text{ contr.}} = \sum_{l=1}^{L} \Psi^{i_1\ldots i_L}\, \mathrm{tr}(\phi^{\mathrm{cl}}_{i_1}\ldots\overbrace{\phi_{i_l}\ldots\phi^{\mathrm{cl}}_{i_L}})\,\mathrm{tr}(Y_1 Y_2^{\mathrm{cl}}) \\ + (Y_1 \leftrightarrow Y_2)\end{aligned} \tag{6.76}$$

[7] The notation differs in some aspects. Most notably, the scalars ϕ denote *real* scalars in the paper but *complex* scalars here and elsewhere in the thesis.

and thanks to the second trace the contraction yields a $(k \times k)$-dimensional matrix

$$[\mathrm{T}_{i_l,Y_1Y_2}]^{s}{}_{s'} = \langle [\phi_{i_l}]^{s}{}_{s'} [Y_1]^{r}{}_{r'} \rangle [Y_2^{\mathrm{cl}}]^{r'}{}_{r} \,. \tag{6.77}$$

This matrix is special as the orthogonality (6.59) and tensor operator property (6.60) of $\hat{Y}_\ell^m$ conspire together with the basis relation (6.61) such that T_{i_l,Y_1Y_2} always can be written in terms of the generators t_i. Further more, it turns out that the contraction is zero whenever $\mathrm{T} \propto t_3$. This makes it possible to view the contraction as an operation on the spin-chain site in the Bethe state $|\Psi\rangle$ that corresponds to $\mathcal{O}_\Psi$. The sum over l in (6.76) turns the site-wise action into the spin-chain operators $S^\pm$ and S^3. We have

$$\begin{aligned} \langle \mathcal{O}_\Psi \mathcal{O}_{Y_1Y_2} \rangle_{1 \text{ contr.}} &= \sum_{l=1}^{L} \Psi^{i_1 \dots i_L} \operatorname{tr}\left(\phi_{i_1}^{\mathrm{cl}} \dots (\mathrm{T}_{i_l,Y_1Y_2} + \mathrm{T}_{i_l,Y_2Y_1}) \dots \phi_{i_L}^{\mathrm{cl}} \right) \\ &\propto \langle \mathrm{MPS} | Q_{Y_1Y_2} | \Psi \rangle \,. \end{aligned} \tag{6.78}$$

The spin-chain operator $Q_{Y_1Y_2}$ contains matrix elements built from the propagators and thus carries the spacetime dependence that we have suppressed in this section. Their precise form are written in Paper I but we note here that the action of $Q_{Y_1Y_2}$ splits into two cases depending on whether the classical values Y_1^{cl} and Y_2^{cl} are equal or not.

- *Case* $Y_1^{\mathrm{cl}} = Y_2^{\mathrm{cl}}$. The Bethe states are eigenstates to $Q_{Y_1Y_2}$ and the two-point function is proportional to the one-point function of $\mathcal{O}_\Psi$:

  $$\langle \mathcal{O}_\Psi \mathcal{O}_{Y_1Y_2} \rangle_{1 \text{ contr.}} = (c^\uparrow (L-M) + c^\downarrow M) \langle \mathcal{O}_\Psi \rangle_{\text{tree}} \,. \tag{6.79}$$

  The (spacetime dependent) coefficients $c^\uparrow$ and $c^\downarrow$ are written in the appendix of Paper I.

- *Case* $Y_1^{\mathrm{cl}} \neq Y_2^{\mathrm{cl}}$. In this case, $Q_{Y_1Y_2}$ induces spin-flips increasing and lowering the spin by one in the two resulting terms. The condition that only unpaired Bethe states may have non-vanishing overlap with the MPS then implies that $\mathcal{O}_\Psi$ must correspond to a Bethe descendant, $|\Psi_{L,M+n}\rangle = (S^-)^n |\Psi_{L,M}\rangle$, where n is odd. Using the overlap (6.22) and the norm (5.26), the result is

  $$\begin{aligned} &\langle \mathcal{O}_{\Psi_{L,M+n}} \mathcal{O}_{Y_1Y_2} \rangle_{1 \text{ contr}} = \\ &\qquad \left(c^+ n (L-2M-n+1) \, C^+_{L,M,n} + c^- \, C^-_{L,M,n} \right) \langle \mathcal{O}_{L,M} \rangle_{\text{tree}} \end{aligned} \tag{6.80}$$

  with

  $$C^\pm_{L,M,n} = \frac{(n \mp 1)! \left(\frac{L}{2} - M \right)!}{\left(\frac{n \mp 1}{2} \right)! \left(\frac{L-2M-n \pm 1}{2} \right)!} \sqrt{\frac{(L-2M-n)!}{n!(L-2M)!}} \,. \tag{6.81}$$

  Again, the coefficients c^+ and c^- carrying the propagators are listed in Paper I.

6.5 β-deformed $\mathcal{N} = 4$ SYM with a defect

The results presented in this chapter inspired the question whether the integrable structure, and in particular the determinant formula for one-point functions, survive in the β-deformed theory. This was the underlying inquiry of Paper II. The results were only partly conclusive and are summarized below in section 6.5.2.

6.5.1 Vacuum solution in the β-deformed theory

The first step is to see whether there is any similar setup at all. Starting from the β-deformed $\mathcal{N} = 1$ superspace action (3.7), it was shown in Paper II that there is indeed a β-deformed vacuum solution in very close analogy to the non-deformed solution in (6.7).

By solving for the auxiliary fields D and F_i and factorizing the kinetic and potential terms for the scalars, it is possible to write the Lagrangian in two negatively semi-definite terms whose simultaneous vanishing yields an extremum. The resulting, so called, BPS equations are

$$[\bar{\phi}_i, \phi^i] = 0, \tag{6.82a}$$

$$-\frac{\partial}{\partial z}\phi^i = \mathrm{i}g\, \varepsilon^{ijk}[\bar{\phi}_j, \bar{\phi}_k]_* \,. \tag{6.82b}$$

Expanding the $*$-commutator and writing them in the real scalar fields φ_i, we find the solution

$$\varphi_i^{\mathrm{cl}} = -\frac{1}{z}\, t_i \oplus \mathbb{0}_{(N-k)}, \qquad i = 1,2,3, \tag{6.83}$$

where again φ_j^{cl}, $j = 4,5,6$, are zero. This is on the form identical to the undeformed solution (6.7) but the matrices t_i now satisfy the q-deformed $SU(2)$-algebra[8]

$$t_i = \mathrm{i}\,\epsilon^{ijk}\,[t_j, t_k]_q \,, \tag{6.84}$$

where the q-commutator is defined as

$$[t_i, t_j]_q = q^{ij}\, t_i t_j - \frac{1}{q^{ij}}\, t_j t_i \tag{6.85}$$

and $q^{ij} = q^{\mathrm{sgn}(ij)}$. Here, $\mathrm{sgn}(ij)$ is $+1$ whenever (ij) are taken in cyclic order from (123) and -1 otherwise. The deformation parameter β is contained in $q = \mathrm{e}^{\frac{\mathrm{i}}{2}\beta}$.

Explicit expressions for a $(k\times k)$-dimensional representation for the $SU(2)_q$-algebra are written in Paper II. But as we move to the spin-chain picture of this setup, we will need some of the properties of t_i.

[8] We keep the same notation t_i and update their definition to deformed generators.

Some properties of the $SU(2)_q$-representation

The representation matrices t_i for the $SU(2)_q$-algebra differ from the undeformed case in an important aspect. The transposition property (6.12) does no longer hold in general. Instead they satisfy

$$\{t_1, t_2, t_3\}^T = \{t_1, -\tilde{V} t_2 \tilde{V}^{-1}, t_3\} \tag{6.86}$$

where the matrix $\tilde{V}$ is diagonal with entries $q^{\big(2(k-2)+6\big)i-2i^2}$, $i = 1, \cdots, k$.

Only the special case $k = 2$ retains the transpositions of (6.12). In fact, they are proportional to the undeformed matrices with

$$t_i = \tfrac{1}{\cos(\beta/2)} t_i = \tfrac{1}{\cos(\beta/2)} \tfrac{1}{2} \sigma_i \,, \qquad (k = 2). \tag{6.87}$$

The similarity transformations U and V from equations (6.20) do however still yield the same relations for the deformed representation, i.e.

$$U t_1 U^{-1} = t_1 \,, \quad U t_{2,3} U^{-1} = -t_{2,3} \,, \quad V t_{1,2} = -t_{1,2} \,, \quad V t_3 V^{-1} = t_3 \,. \tag{6.88}$$

6.5.2 Spin-chain picture and the β-deformed MPS

As above, we now restrict our attention to the two scalar-subsector[9] and study the same set of operators.

The transition to the spin-chain picture is entirely analogous to the undeformed discussion above, with the spin-chain now having a twist such that the Hamiltonian is given by (5.55). However, there are some significant differences regarding the MPS.

First and foremost, the definition (5.48) of an integrable MPS is no longer relevant in the twisted spin-chain. The reason is that the behavior (5.46) of the charges under parity is changed to

$$\Pi Q_n \Pi = (-1)^n \, Q_n \Big|_{\beta \to -\beta}, \tag{6.89}$$

such that the odd charges no longer annihilate the MPS. The selection rule that singles out unpaired Bethe states is hence missing and, as of yet, no replacing condition has been found. This naturally breaks any hope to transfer the determinant formula from the undeformed theory directly.

Equation (6.89) also makes the β-deformed MPS irrelevant for the direct construction of MPSs in [68]. These rely on the parity behavior of the undeformed theory.

The broken $SU(2)$-symmetry of the spin-chain also obstructs the avenue to study integrability properties of the MPS via the Neél state, as also mentioned in section 5.4.1.

[9] With the broken R-symmetry, the nomenclature of an $SU(2)$-subsector is no longer strictly correct.

What remains is however the selection rule of even L and M since the same similarity argument goes through, thanks to equations (6.88). The MPS seems[10] to be parity invariant and that it is possible to find a similarity transformation in the auxiliary space such that

$$\Pi\mathcal{T}(u)\Pi|\mathrm{MPS}\rangle = \mathcal{T}(u)|\mathrm{MPS}\rangle\,. \tag{6.90}$$

With these remarks on the lack of a determinant formula, some additional results of the simplest overlaps were presented in Paper II.

Simple examples of overlaps

Without a general formula, we cannot make a full comparison with the undeformed theory. The simplest overlaps are however enough to highlight some interesting differences when the β is introduced.

The most striking change is the scaling with k, which we saw in equation (6.41) was k^{L-M+1} for the undeformed overlaps. For any non-zero β, this is changed to an underlying linear scaling for all the considered examples, i.e. for $M = 0, 2$ with general L and $L = 8, M = 4$.

Spin-chain vacuum.

The operator $\mathrm{tr}(\phi_1^L)$ was found to have the overlap

$$\begin{aligned}\langle \mathrm{MPS}|\Omega\rangle = \mathrm{tr}(t_3^L) = \sum_{i=1}^{k} d_{k,i}^L &= \frac{2}{(\sin\beta)^L}\sum_{i=1}^{\lfloor k/2\rfloor}\left[\sin\tfrac{\beta(k+1-2i)}{2}\right]^L \\ &= \frac{1}{(2\sin\beta)^L}(-1)^{\frac{L}{2}}\sum_{m=0}^{L}(-1)^m\binom{L}{m}\frac{\sin\frac{\beta k(L-2m)}{2}}{\sin\frac{\beta(L-2m)}{2}}\,,\end{aligned} \tag{6.91}$$

with the k-scaling

$$\langle \mathrm{MPS}|\Omega\rangle \overset{k\to\infty}{\sim} \frac{1}{(2\sin\beta)^L}\binom{L}{L/2}k + \cdots \tag{6.92}$$

For this state, the overlap has purely linear k-asymptotics.

Two excitations.

The states with $M = 2$ have of course a more complicated expression and also a more intricate scaling with k. The overlap reads

$$\begin{aligned}\langle \mathrm{MPS}|\Psi_{M=2}\rangle = \frac{1}{2}\frac{1}{(\sin\beta)^L}\sum_{i=1}^{k-1}\Bigg[&\left(1-\frac{\cos\beta+\cos\beta k}{\cos\beta+\cos[\beta(k-2i)]}\right)\left(\sin\tfrac{\beta(k+1-2i)}{2}\right)^{L-2} \\ &\times\left(S(p)\,Y_i(L,k,\beta,p)+\overline{Y_i(L,k,\beta,p)}\right)\Bigg]\end{aligned} \tag{6.93}$$

[10] A proof for $k > 2$ is missing but numerics indicates that it holds for all k.

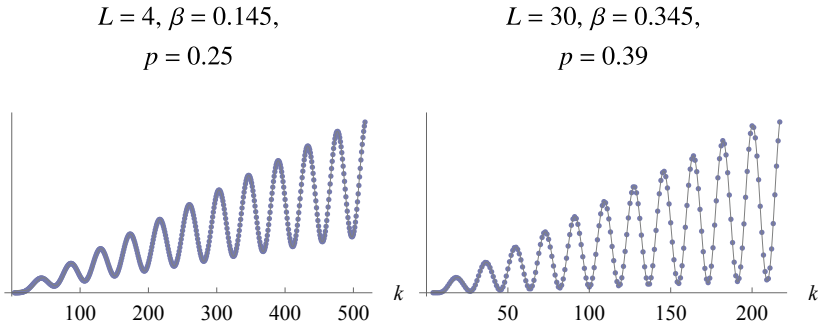


Figure 6.3. The scaling with k of the overlap between the MPS and two examples states with $M = 2$. The absolute values of the overlaps are plotted as dots and the gray line is fitted from an ansatz written in paper II.

where we have packed the following into the symbols

$$Y_i = \mathrm{e}^{\mathrm{i}(p+\beta)} \frac{(L-1) - L\mathrm{s}_i + (\mathrm{s}_i)^L}{(1-\mathrm{s}_i)^2}\,, \qquad \mathrm{s}_i = \mathrm{e}^{\mathrm{i}(p+\beta)} \frac{\sin\frac{\beta(k-1-2i)}{2}}{\sin\frac{\beta(k+1-2i)}{2}}. \tag{6.94}$$

The bar denotes complex conjugation and the S-matrix reduces to

$$S(p) = -\frac{1 + \mathrm{e}^{2\mathrm{i}\beta} - 2\,\mathrm{e}^{-\mathrm{i}p}}{1 + \mathrm{e}^{2\mathrm{i}\beta}(1 - 2\,\mathrm{e}^{\mathrm{i}p})} \overset{\text{(BAE)}}{=} \mathrm{e}^{-\mathrm{i}L(p+\beta)} \tag{6.95}$$

due to the translational invariance condition

$$p = p_1 = -p_2 - 2\beta\,. \tag{6.96}$$

The k-scaling is an oscillation, with approximate period of βk, times and underlying linear dependence. Some example plots can be seen in figure 6.3.

For further details, and the numerical analysis of the overlaps with $L = 8, M = 4$, we refer to paper II.

7. Quantum Spectral Curve for the β-deformation

We discussed three different formalisms for obtaining the spin-chain spectrum in chapter 5. These have been exploited for the 1-loop calculations but are not sufficient for higher perturbative orders of anomalous dimensions in $\mathcal{N} = 4$ SYM. The push towards higher and higher number of accounted interaction loops has gone through several methods, starting with the asymptotic Bethe ansatz based on S-matrix analysis [30, 89], supplemented with Lüscher corrections to account for finite-size effects[1] [90–96]. The thermodynamic Bethe ansatz (TBA) combined the analysis of bulk and finite size and could further be refined into the so called Y-system, T-system and the finite set of non-linear integral equations of FiNLIE [97–117]. The most recent and refined method of them is the Quantum Spectral Curve, which captures the information of the TBA T-system in only a finite number of Q-functions [2, 3].

This chapter provides a minimal introduction to this formalism and a summary of the applications in Paper IV. For full accounts, see the provided references and [4–6].

The name "spectral curve" stems from a construction in the string worldsheet σ-model on the string theory side of the AdS/CFT correspondence, first made in [118, 119]. Strings are one-dimensional objects and as they propagate through spacetime, they sweep out two-dimensional surfaces, their worldsheets, parametrized by the two coordinates τ and σ.

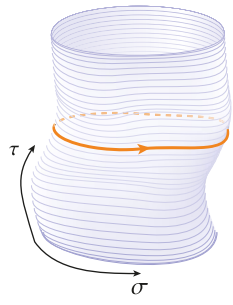
It is possible to define a so called Lax connection on the worldsheet. It is a super-Lie algebra valued one-form which depends on the complex spectral parameter u and satisfies the flatness condition

$$\partial_\alpha \mathcal{L}_\beta(u) - \partial_\beta \mathcal{L}_\alpha(u) - [\mathcal{L}_\alpha(u), \mathcal{L}_\beta(u)] = 0\,, \qquad \alpha,\beta \in \{\tau,\sigma\}\,. \tag{7.1}$$

From this object we can construct the monodromy

$$\mathcal{M}(u;\tau) = \mathrm{Pexp}\left[\int_0^{2\pi} \mathrm{d}\sigma\, \mathcal{L}_\sigma(u;\tau,\sigma)\right] \tag{7.2}$$

[1] The S-matrix approach starts out from infinite volume $L \to \infty$ and the asymptotic Bethe ansatz is only valid below the wrapping order.



$$\mathrm{Pexp}\left[\int_0^{2\pi} \mathrm{d}\sigma \mathcal{L}_\sigma(u;\tau,\sigma)\right]$$

Figure 7.1. The worldsheet monodromy (7.2) is a path ordered integral of the σ-component of the Lax operator around a closed path wrapping the worldsheet once.

by a path-ordered integral along a closed path that wrap around the worldsheet once, see figure 7.1. The flatness condition (7.1) ensures that worldsheet time translations simply amount to a similarity transformation of $\mathcal{M}$, i.e.

$$\partial_\tau \mathcal{M} = U \mathcal{M} U^{-1}\,, \tag{7.3}$$

such that its eigenvalues λ are conserved. As eigenvalues, they satisfy the characteristic equation

$$\det\Big(\mathcal{M}(u) - \lambda(u)\Big). \tag{7.4}$$

It follows that $\lambda(u)$ is a multi-valued function of u, or a single-valued function on a multi-sheeted Riemann surface called the spectral curve.

Diagonalizing $\mathcal{M}$, we can parametrize the eigenvalues as

$$\mathcal{M}(u) = \mathrm{diag}\Big(\mathrm{e}^{\mathrm{i}p_1(u)}, \ldots, \mathrm{e}^{\mathrm{i}p_4(u)} \,\Big|\, \mathrm{e}^{\mathrm{i}p_{\hat{1}}}(u), \ldots, \mathrm{e}^{\mathrm{i}p_{\hat{4}}(u)}\Big) \tag{7.5}$$

where the eight p are called quasi-momenta and have more preferable analytic structures than $\lambda(u)$. They form an eight-sheeted Riemann surface connected by branch cuts with a structure inherited from (7.3). $p_i(u)$ are specified through their large u-asymptotics, the periods around and through the cuts and their action-variables, i.e. their integrals of u around the cuts. The asymptotics are controlled by the string state weights of the symmetry group, including the string energy.

The QSC can be viewed as the quantization of this spectral curve. Its basic structure is a Q-system of 256 Q-functions which are all connected through QQ-relations. In the QSC, the Q-functions are now multi-valued functions of the spectral parameter while the limit $g \to 0$ corresponds to the spin-chain Q-system we touched upon in chapter 5.

The QSC has been developed both with and without twists. It has been successfully used to calculate the anomalous dimensions perturbatively up to 11 loops analytically and to an incredible accuracy numerically, thus far outperforming all other known methods [5, 120].

Solutions to the QSC

Each single-trace operator in $\mathcal{N} = 4$ SYM corresponds to a solution to the QSC, which as stated contains the anomalous dimension of the associated multiplet. A large number of such solutions have been found perturbatively and there is now plenty of explicit data in the literature, as well as many other applications of the QSC [4, 5, 120–133].

The introduction of twists breaks the multiplets and naturally alter the solution space of the QSC. Some of this is already known through other techniques. For the β-deformation, these include regular QFT-calculations which gave the complete 1-loop dilation operator and the four-loop anomalous dimensions of one-magnon of two-magnon $SU(2)$ states in [134, 135], and for one-magnon states to the first wrapping order in [136]. The TBA equations and Y-system were generalized to the β-deformation in [137] and [138] while four loop asymptotic Bethe ansatz calculations with Lüscher corrections were done for length-2 and length-3 operators in the $\mathfrak{sl}(2)$ sector in [139]. See also results in [140, 141].

However, there is as of yet not as many known explicit results by use of the twisted QSC and it was the goal of Paper IV to initiate such explicit analytical applications of the QSC for the β-deformation of $\mathcal{N} = 4$ SYM.

We saw in section 3.1 how the β-deformation killed several of the symmetry generators. The parts of the $\mathcal{N} = 4$ SYM multiplets connected by these generators thus split up. Still, some of the symmetries remain which leads to submultiplets, as it was called in Paper IV. The full structure of this splitting is an interesting problem to work out. The scope of Paper IV was however less ambitious; it aimed to find the leading solutions to a number of operators that join into the Konishi multiplet at $\beta = 0$, to perturbatively calculate their anomalous dimensions and to display the features and challenges in the explicit application of QSC to the β-deformed $\mathcal{N} = 4$ SYM. The results are briefly summarized in section 7.5 while we in the following will go through how to get there.

7.1 The Q-system

We have already had a peek at some QQ-relations towards the end of chapter 5 and we now return to them.

The full set of Q-functions of an integrable system and their QQ-relations constitute a Q-system. Our example for the $SU(2)$-chain had $2^2 = 4$ Q-functions. The full symmetry group $PSU(2,2|4)$ of $\mathcal{N} = 4$ SYM has a total of $2^8 = 256$ Q-functions stemming from all possible duality transformations.

To organize them, we label each Q-function as in section 5.7.1, with two anti-symmetric multi-indices $A,I \supseteq \{1, 2, 3, 4\}$, e.g.

$$Q_{12|134} = -Q_{21|134} = -Q_{12|143}\,. \tag{7.6}$$

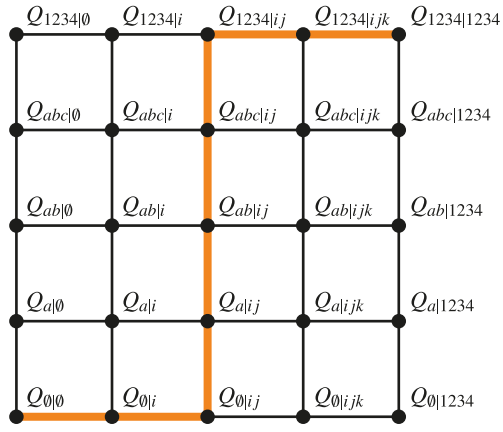


Figure 7.2. The $GL(4|4)$ Q-system is a lattice of 5×5 nodes where the 256 Q-functions sit, arranged according to their number of indices. The grading can be indicated directly on this lattice, just as it was done in figure 2.2. Here the grading is exemplified by **2222**.

It is further very practical to combine the pictures of the grading path in figure 2.2 and the Dynkin diagram and place them all into a 5×5 lattice, as in figure 7.2. [4] developed a method for the untwisted QSC to extend this with Young diagrams for the representations of the solution. We will, however, stick to this 5×5 lattice.

The Q-functions satisfy two types of QQ-relations: bosonic that involve Q-functions on a line and fermionic that involves Q-functions from all corners of a single quadrant in the lattice:

$$\text{bosonic:} \qquad Q_{A|I}Q_{Aab|I} = W(Q_{Aa|I}, Q_{Ab|I})\,, \tag{7.7a}$$

$$Q_{A|I}Q_{A|Iij} = W\Big(Q_{A|Ii}, Q_{A|Ij}\Big)\,, \tag{7.7b}$$

$$\text{fermionic:} \qquad Q_{Aa|I}Q_{A|Ii} = W(Q_{Aa|Ii}, Q_{A|I})\,, \tag{7.7c}$$

where the Wronskian is defined as

$$W(A,B) = A^{+}B^{-} - A^{-}B^{+}\,. \tag{7.7d}$$

They are also written and illustrated in figure 7.3.

Knowing all about the Q-system is, among other things, to know the entire spectrum. However, the full system is a very redundant description. In fact, we can generate the whole system from only knowing the single-index functions

$$\mathbf{P}_a := Q_{a|\emptyset}\,, \qquad\qquad \mathbf{Q}_i := Q_{\emptyset|i}\,, \tag{7.8}$$

by sequential use of the QQ-relations in figure 7.3. These eight functions are the analogue of the eigenvalues p in (7.5). The procedure is thoroughly explained in either of the references [3, 142].

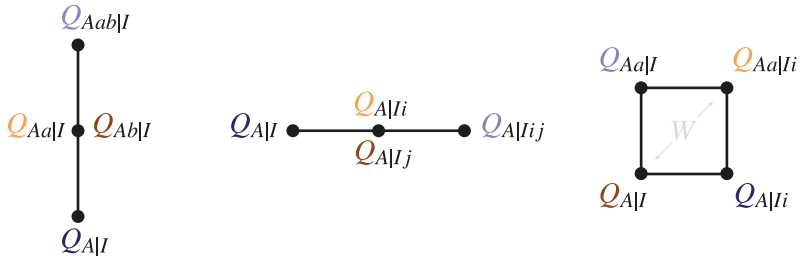


Figure 7.3. The two bosonic and the fermionic type of QQ-relations of equation 7.7 relate the Q-functions at the respective nodes according to $QQ = Q^{+}Q^{-} - Q^{-}Q^{+}$.

7.1.1 A dual system

There is a symmetry of the Dynkin diagram which we broke with our choice of labeling of the Q-functions, the one of turning it upside-down. To incorporate this symmetry in the construction, we can define a Hodge dual system with the functions

$$Q^{A|I} := (-1)^{|A'||I|}\varepsilon^{A'A}\varepsilon^{I'I}Q_{A'|I'}\,, \qquad \text{(no sum over } A' \text{ and } I') \tag{7.9}$$

where the set of Q-functions $Q^{A|I}$ satisfies the same QQ-relations as the lower index ones. Here $|A|$ denotes the number of indices within the multi-index A.

7.1.2 Asymptotics

The asymptotics of the Q-functions are given by the quantum numbers of the HWS operator which thus specifies its corresponding solution. This is where the anomalous dimensions enter into the formalism. The quantum numbers λ_a related to R-symmetry do naturally not involve γ, but the AdS-related quantum numbers do and carry it according to

$$\nu_i = \nu_i\big|_{g=0} + \tfrac{1}{2}\{-\gamma, -\gamma, +\gamma, +\gamma\}_i\,. \tag{7.10}$$

The actual quantum numbers of $PSU(2,2|4)$ are however the differences $\lambda_a - \lambda_{a+1}$ and $\nu_i - \nu_{i+1}$ which allows the freedom of an unspecified shift Λ into these numbers. This is related to what is called a gauge symmetry of the Q-system.

The asymptotics are somewhat different in the twisted and untwisted formulations of the QSC. For a full account of this, see [6]. Here we just exemplify them for the single-index function (7.8) in the β-deformation. They are expressed in terms of

$$\hat{\lambda}_a = \lambda_a - \sum_{b<a}\delta_{\mathrm{x}_a,\mathrm{x}_b} + \sum_{i<a}\delta_{\mathrm{x}_a,1} + \Lambda\,, \tag{7.11}$$

$$\hat{\nu}_i = \nu_i - i + 1 + \sum_{a<i}\delta_{\mathrm{x}_a,1} - \Lambda\,, \tag{7.12}$$

where we have introduced the twist factors

$$\mathsf{x}_a = \left\{ e^{\mathrm{i}\beta(n_{\mathbf{f}_2} - n_{\mathbf{f}_3})}, e^{\mathrm{i}\beta(n_{\mathbf{f}_3} - n_{\mathbf{f}_1})}, e^{\mathrm{i}\beta(n_{\mathbf{f}_1} - n_{\mathbf{f}_2})}, 1 \right\}_a . \tag{7.13}$$

The β-deformed asymptotics for $\mathbf{P}_a$ and $\mathbf{Q}_i$ are then

$$\mathbf{P}_a \sim \mathsf{A}_a \mathsf{x}_a^{\mathrm{i}u} u^{-\hat{\lambda}_a} , \tag{7.14a}$$

$$\mathbf{P}^a \sim \mathsf{A}^a \mathsf{x}_a^{-\mathrm{i}u} u^{\hat{\lambda}_a - 4\delta_{\mathsf{x}_a,1} + \sum_{b\neq a}\delta_{\mathsf{x}_a,\mathsf{x}_b}} , \tag{7.14b}$$

$$\mathbf{Q}_i \sim \mathsf{B}_i u^{-\hat{\nu}_i} , \tag{7.14c}$$

$$\mathbf{Q}^i \sim \mathsf{B}^i u^{\hat{\nu}_i - \sum_a \delta_{\mathsf{x}_a,1} + 3} . \tag{7.14d}$$

The normalizations A and B also depend on the quantum numbers and are subjected to rather strict conditions, though allow for some freedom of choice.

7.2 The $\mathbf{P}\mu$-system

As stated, the full Q-system carries a lot of redundancy. There exists, however, a very slim formulation that captures all of the information in only 14 functions: the 2×4 functions $\mathbf{P}$ and six functions μ_{ab}, with a,b anti-symmetric. The system is accordingly called the $\mathbf{P}\mu$-system and is very well-suited for perturbative calculations.[2]

The primary idea is to use the precise analytic structures of these functions and a key difference equation (eq. (7.23) below). In fact, this equation is also solved by the central Q-functions $Q_{ab|ij}$ and, as a consequence, μ_{ab} have to be linear combinations of them. μ control the analytic structure of $\mathbf{P}$ by relating their values on different sheets. Let us make that statement a bit more precise after presenting their individual structures.

Analytic structure of P

$\mathbf{P}$, being just the single-indexed Q-functions (7.8), are multi-valued functions of u. In the short cut formulation, $\mathbf{P}$ has one Riemann sheet with a single branch-cut between the branch-points $\pm 2g$. The location of the branch-points is the only place in the QSC where the coupling g appears explicitly. We denote the values on the first Riemann sheet as $\mathbf{P}$ and the those on the second sheet as $\tilde{\mathbf{P}}$. This second sheet has an infinite number of cuts separated by i between each pair of points $\pm 2g + \mathrm{i}\mathbb{Z}$. See the left hand side of figure 7.4.

Analytic structure of μ

μ has an infinite number of cuts on all its sheets but the first and the second sheets are intimately connected. The values on the second sheet is the first sheet shifted by i, that is to say

$$\mu^{[2]} = \tilde{\mu} . \tag{7.15}$$

[2] There are also other compact formulations in the litterature, see e.g. [120].

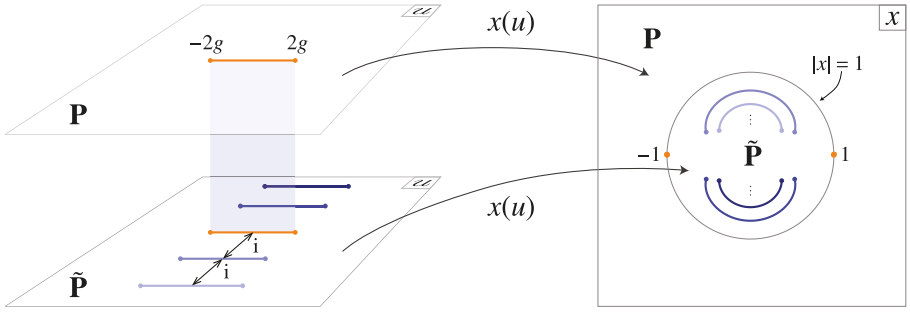


Figure 7.4. The analytic structure of **P**. To the left, there is a single cut on the first Riemann sheet in the u-plane between the points $\pm 2g$ and an infinite number of cuts on the second sheet at $\pm 2g + \mathrm{i}\mathbb{Z}$. The single cut is dissolved by the Zhukowsky map $x(u)$ which maps the first sheet, and hence **P**, to the region outside the unit circle while the second sheet, $\tilde{\mathbf{P}}$, is mapped into the interior. (The plot uses an example value of $g = 2$.)

Importantly, it follows that both the expressions

$$\mu + \mu^{[2]} \qquad \text{and} \qquad \frac{\mu - \mu^{[2]}}{\sqrt{u^2 - 4g^2}} \tag{7.16}$$

are regular on the real axis.

Furthermore, QSC solutions that correspond to single-trace operators impose powerlike asymptotics on μ, up to exponential factors of the twist, i.e.

$$\mu_{ab} \sim (\mathsf{x}_a \mathsf{x}_b)^{\mathrm{i}u} \cdot u^{M_{ab}} \tag{7.17}$$

for some number M_{ab}. The zero-momentum condition translates to the requirement that

$$\lim_{u \to 0} \frac{\mu_{ab}}{\mu^{[2]}_{ab}} = 1\,. \tag{7.18}$$

Relations

The functions μ connect the values of **P** on its two first sheets

$$\tilde{\mathbf{P}}_a = \mu_{ab} \mathbf{P}^b\,, \qquad \tilde{\mathbf{P}}^a = \mu^{ab} \mathbf{P}_b\,. \tag{7.19}$$

Here we have introduced the upper-index μ^{ab} which is related to the lower-index μ_{ab} through

$$\mu^{ab} = -\frac{1}{2} \varepsilon^{abcd} \mu_{cd} \frac{1}{\mathrm{Pf}(\mu)}\,, \qquad \mathrm{Pf}(\mu) = \frac{1}{8} \varepsilon^{abcd} \mu_{ab} \mu_{cd}\,. \tag{7.20}$$

The Pfaffian $\mathrm{Pf}(\mu)$ can be shown to be a constant that depends on the normalization; it will enter as a g-expansion with determinable coefficients in the perturbative calculations.

Since all branch-cuts are of square-root type and we get back to the starting point by passing twice, (7.19) only makes sense if

$$\mu_{ab}\mu^{bc} = \delta_a^c \tag{7.21}$$

which indeed is the case.

Moreover, $\mathbf{P}$, $\tilde{\mathbf{P}}$ and μ are related through

$$\mu_{ab} - \tilde{\mu}_{ab} = -\mathbf{P}_a\tilde{\mathbf{P}}_b + \mathbf{P}_b\tilde{\mathbf{P}}_a \tag{7.22}$$

which thanks to the analytic relations becomes a difference equation for μ,

$$\mu_{ab} - \mu_{ab}^{[2]} = -\mathbf{P}_a\mathbf{P}^c\mu_{bc}^{[1\pm1]} + \mathbf{P}_b\mathbf{P}^c\mu_{ac}^{[1\pm1]}. \tag{7.23}$$

This equation is the main focus of the perturbative calculations in the $\mathbf{P}\mu$-system.

The Zhukowsky map

The Zhukowsky variable x is ubiquitous in the context of perturbative calculations in $\mathcal{N} = 4$ SYM and its deformed cousins. It is a two-valued map of the spectral parameter,

$$x + \frac{1}{x} = \frac{u}{g}. \tag{7.24}$$

As such, we always denote x for the larger value $|x| > 1$ and replace $x \to x^{-1}$ for the smaller.

A crucial feature of the Zhukowsky map is that the single branch-cut on the first sheet of $\mathbf{P}$ is mapped onto the unit circle in the x-plane. It separates the first sheet, which is mapped to the outside region $|x| > 1$, from the second sheet which is mapped onto the interior $|x| < 1$. Hence, the cut is in effect dissolved, a fact that can be exploited when making an ansatz for $\mathbf{P}$ and $\tilde{\mathbf{P}}$. An illustration of the map is to be found to the right in figure 7.4.

7.3 Leading order Q-system

The starting point for the perturbative calculations is the leading solution in g to the Q-system. It can be obtained through making an ansatz for a subset of the Q-functions, called distinguished Q-functions. There is one of them at each node in the Q-system and they are, up to overall exponentials and known prefactors, all polynomials. We can thus label them with only two indices $a = |A|$ and $s = |I|$ and choose the labeling of all the Q-functions such that

$$\mathbb{Q}_{a,s} := Q_{12\ldots a|12\ldots s}, \tag{7.25}$$

e.g. $\mathbb{Q}_{2,3} = Q_{12|123}$. The distinguished Q-functions are all related by the fermionic QQ-relations in figure 7.3. Specifically,

$$\mathbb{Q}_{a+1,s} = \frac{\mathbb{Q}^{+}_{a+1,s+1}\mathbb{Q}^{-}_{a,s} - \mathbb{Q}^{-}_{a+1,s+1}\mathbb{Q}^{+}_{a,s}}{\mathbb{Q}_{a,s+1}}. \tag{7.26}$$

The ansatz relies on the structure

$$\mathbb{Q}_{a,s} = \left(\prod_{\tilde{a}=1}^{a} \mathsf{x}_{\tilde{a}}^{\mathrm{i}u}\right) f_{a,s}\, q_{a,s} \tag{7.27}$$

where $f_{a,s}(u)$ are simple and known rational functions of u called fusion factors and $q_{a,s}(u)$ are polynomials.

A technique to choose the number $M_{a,s}$ of roots of these polynomials, based on Young diagrams, was developed for the untwisted case in [4]. Paper IV proposed an adaption to the β-deformation inspired from this method, in which the numbers of roots along the grading path stay the same as for $\beta = 0$. The numbers of roots for the rest of the distinguished Q-functions can then be inferred from the QQ-relations, paying attention to the fusion factors and to which QQ-relations that involve exponential twist factors.

An ansatz can thus be made for $\mathbb{Q}_{a,s}$ on the grading path

$$\mathbb{Q}_{a,s} = \left(\prod_{\tilde{a}=1}^{a} \mathsf{x}_{\tilde{a}}^{\mathrm{i}u}\right) f_{a,s}(u) \left(u^{M_{a,s}} + \sum_{k=0}^{M_{a,s}-1} u^{k} c_{a,s;k}\right) \tag{7.28}$$

and all other Q-functions can be generated by the fermionic QQ-relations (7.26). In that process, we can use our knowledge of the structure (7.27) to collect a number of conditions to fix the ansatz coefficients c_k. Dividing by the exponentials and fusion factors, we know that the QQ-relations should give us polynomials in u and we define the $q_{a,s}$ of $\mathbb{Q}_{a,s}$ to be the quotient from this polynomial division, while the remainders need all to vanish. The collection of all such zero-remainder conditions fixes all the ansatz coefficients.

Once the numbers of roots are determined on the Q-system, it is possible to make the starting ansatz on any path from $\mathbb{Q}_{\emptyset|\emptyset}$ to $\mathbb{Q}_{1234|1234}$, preferably minimizing the number of introduced constants $c_{a,s;k}$.

Notice that this procedure immediately gives the polynomials with the Bethe roots as zeros and the described algorithm is, in fact, a very efficient way of finding the solutions to the 1-loop Bethe equations.

7.3.1 From $\mathbb{Q}_{a,s}$ to $\mathbf{P}\mu$

With knowledge of the leading distinguished Q-functions, we can start building other leading Q-functions, in particular

$$Q_{a|\emptyset}, Q_{\emptyset|i}, Q_{a|i}, Q_{ab|ij}, Q_{abc|ijk}, Q_{abc|1234}, \tag{7.29}$$

which are the ones needed for an efficient route to the leading $\mathbf{P}\mu$-system. The procedure is explained in [4] and briefly summarized in paper IV.

7.4 A perturbative algorithm

The real power of the $\mathbf{P}\mu$-system shines through in the perturbative calculations. Vital to the algorithm is an ansatz for $\mathbf{P}$ expressed in the Zhukowsky variable. The salient feature of this map is that the expansions in g of the ansatz is convergent, not only for $|x| > 1$, but also in a finite region within the unit circle[3]. That means that the ansatz for $\mathbf{P}$ also provides an ansatz for $\tilde{\mathbf{P}}$ close to the unit circle. A finite number of coefficients are needed at each order and, using the different relations and analytic requirements, it is possible to fix them all, order by order.

The algorithm consists of five steps carried out at each order, starting from the input of the leading Q-system and the first order ansatz for $\mathbf{P}$ and $\tilde{\mathbf{P}}$. All objects are written in terms of their expansions $X = \sum_n g^{2n} X^{(n)}$.[4]

The steps are explained with detailed examples in both [5] and paper IV so we only state them as an overview here:

Step 1 Make the ansatz of Paper IV for $\mathbf{P}^{(n)}$, introducing a number of coefficients to be fixed. Notably, the normalizations A and B contain the anomalous dimension through their dependence on $\hat{\nu}_i$ and its expansion in g thus enters among the constants to fix.

Step 2 Solve the n-loop order part of the difference equation (7.23) to get $\mu_{ab}^{(n)}$. This is done through rewriting equation (7.23) as an inhomogeneous difference equation

$$\nabla\mu_{ab}^{(n)} = -\mathbf{P}_a^{(0)}\mathbf{P}^c_{(0)}\mu_{bc}^{(n)[1\pm1]} + \mathbf{P}_b^{(0)}\mathbf{P}^c_{(0)}\mu_{ac}^{(n)[1\pm1]} + U_{ab}^{(n)} \tag{7.30}$$

where all lower, and hence known, orders of μ are gathered into the source term $U_{ab}^{(n)}$. The difference operator is defined as $\nabla X = X - X^{[2]}$ and has the inverse $\Psi(\nabla X) = X + \mathcal{P}$, with $\mathcal{P}$ being some i-periodic function.

It can be shown [142], that the solution can be obtained as

$$\mu_{ab}^{(n)} = \frac{1}{4} Q^{(0)\ [-1]}_{ab|ij}\, \Psi\left(Q_{(0)}^{cd|ij[-1]} U_{cd}^{(n)} \right) \tag{7.31}$$

[3] More specifically, it is convergent all the way until the outermost of the infinite number of cuts from the second u-sheet. See figure 7.4.

[4] There are solutions that involve also odd powers of g but those are not relevant for our scope. See e.g. [5].

where $\mu_{ab}^{(n)}$ hence contain the constants introduced in step 1 through the source term.

The action of Ψ only (re)produces exponential twist factors, polynomials in u, shifted inverse powers $(u+ni)^{-m}$ and Hurwitz η-functions. These are all algebraic rings which ensures that all occurring expressions can be written as four factor terms, with each factor exclusively belonging to one of these function sets. The Ψ-operation is hence closed. More details on Ψ can be found in [5], Paper IV or in the appendix of [142].

Step 3 Impose the regularity conditions on μ, i.e. expand the expressions (7.16) to order g^{2n} and demand that all poles vanish. This fixes a lot of the coefficients from step 1.

Step 4 Define $\tilde{\mathbf{P}}^{(n)}$ from the relations (7.19). In this process it is also required to expand the Pfaffian $\mathrm{Pf}(\mu) = \sum_n g^{2n} \mathrm{Pf}(\mu)^{(n)}$ as an additional coefficient to fix (the lower orders have already been fixed).

Step 5 Lastly, compare the obtained expressions of $\tilde{\mathbf{P}}^{(n)}$ in step 4 with the general ansatz and fix the remaining coefficients entering at order g^{2n}.

This procedure normally fixes all coefficients such that all objects $X^{(m\leq n)}$ can be used in the next iterative step for order $g^{2(n+1)}$.

7.5 Results and challenges for β-deformed Konishi multiplet

Paper IV aimed to adapt and apply the method to find the leading solution and the following perturbative algorithm for the $\mathbf{P}\mu$-system to the β-deformed Konishi multiplet. Eight different operators were considered and their anomalous dimensions were calculated to various loop orders through a *Mathematica*-implementation. The computational success was, however, strongly solution dependent and the obtained results varied from the eight-loop anomalous dimension for the simplest operators while only a two-loop result could be reached for the most challenging one. The complications met were computationally technical in nature, rather than conceptual, and consisted of either twist-dependent square-roots in the leading solutions or quickly growing rational expressions in x. The latter occurred particularly for the example operator with all x_a different.

The computational performance reflected these complications. In general, all computation times scaled roughly exponentially with the loop order, a pat-

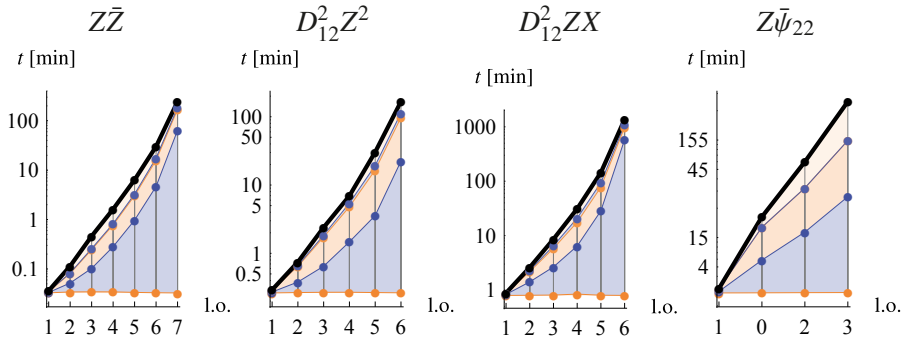


Figure 7.5. The scaling of computation times for four example operators, here labeled as in Paper IV with their potential field content. The $Z\bar{Z}$ to the left has no twist, the operator $D_{12}^2Z^2$ has a simple solution, the D_{12}^2ZX contains $\sqrt{6}$ and $Z\bar{\Psi}_{22}$ contains the more complicated square root $\sqrt{x(2x^2+5x+2)}$. The total computation time is logarithmically plotted as the thick black line. The fractional computation times (in per cent) for the individual steps in the perturbative algorithm are indicated with the shaded colored regions and are independent of the scale on the y-axis. The computation times and reached loop orders are heavily affected by the presence of the square roots.

tern that held for all the considered solutions. The comparison between solutions, though, depends on the details of the involved expressions. An illustration can be found in figure 7.5 where the computation times are plotted for operators without twist, with twist and a simple solution, with a simple square-root in the solution and, finally, with a complicated square-root. As in Paper IV, we label the solutions with a term of possible field content, in this case $Z\bar{Z}$, $D_{12}^2Z^2$, D_{12}^2ZX and $Z\bar{\psi}_{22}$.

As an example of the results for the anomalous dimensions, we restate the eight-loop expansion for the operator with the field content $\psi_{11}F_{11}$:

$$\gamma^{(1)} = 12 \tag{7.32}$$

$$\gamma^{(2)} = -48$$

$$\gamma^{(3)} = -12(c_1 - 29)$$

$$\gamma^{(4)} = -192\zeta_3(c_1 - 4) + 348c_1 - 1440\zeta_5 - 2844$$

$$\gamma^{(5)} = 96\zeta_3(29c_1 + 43) + 2880\zeta_5(c_1 - 4) - 7380c_1 - 5184\zeta_3^2 + 30240\zeta_7 + 22548$$

$$\begin{aligned}\gamma^{(6)} = {} & 6912\zeta_3^2(c_1 - 4) - 44832\zeta_3 c_1 + 192\zeta_3 c_2 + 288\zeta_5(539 - 149c_1) \\ & + 336\zeta_7(334 - 109c_1) + 136428c_1 - 156c_2 \\ & + 155520\zeta_5\zeta_3 - 218016\zeta_3 - 489888\zeta_9 - 143952\end{aligned}$$

$$\begin{aligned}\gamma^{(7)} = {} & 576\zeta_3^2(-17c_1 + 3c_2 - 718) + 597984\zeta_3 c_1 - 7824\zeta_3 c_2 \\ & + 576\zeta_5\zeta_3(863 - 149c_1) - 48\zeta_5(-13586c_1 + 95c_2 + 18267) \\ & + 15120\zeta_7(30c_1 - 113) + 451584\zeta_9 c_1 - 2315988c_1 + 9984c_2 + 124416\zeta_3^3\end{aligned}$$

$$
\begin{aligned}
&-1935360\zeta_7\zeta_3+4639920\zeta_3-993600\zeta_5^2-1287072\zeta_9+7318080\zeta_{11}+170964\\
\gamma^{(8)} = &-\frac{684288}{5}Z_{11}^{(2)}+27648\zeta_3^3(5\mathrm{c}_1+67)-1085184\zeta_3^2\mathrm{c}_1-71136\zeta_3^2\mathrm{c}_2-5217888\zeta_3\mathrm{c}_1\\
&+248112\zeta_3\mathrm{c}_2-1728\zeta_5\zeta_3(-233\mathrm{c}_1+35\mathrm{c}_2+2718)-4032\zeta_7\zeta_3(1573-28\mathrm{c}_1)\\
&-11520\zeta_5^2(11\mathrm{c}_1-218)-9759936\zeta_5\mathrm{c}_1+188400\zeta_5\mathrm{c}_2-8016816\zeta_7\mathrm{c}_1+92856\zeta_7\mathrm{c}_2\\
&-4287840\zeta_9\mathrm{c}_1-5575680\zeta_{11}\mathrm{c}_1+36845004\mathrm{c}_1-384636\mathrm{c}_2-3255552\zeta_5\zeta_3^2\\
&+9091296\zeta_3^2+23224320\zeta_9\zeta_3-78527184\zeta_3-10106928\zeta_5+22256640\zeta_5\zeta_7\\
&+29792664\zeta_7+13615584\zeta_9+\frac{93807648\zeta_{11}}{5}-106007616\zeta_{13}+17947824\,,
\end{aligned}
$$

Here, ζ_i denote multiple zeta values (MZV), also formulated into the single-valued MZV $Z_{11}^{(2)} = -\zeta_{3,5,3}+\zeta_3\zeta_{3,5}$, and we have used the abbreviation $\mathrm{c}_n = \cos(n\beta)$.

For the other calculated anomalous dimensions and leading distinguished Q-functions, we refer to Paper IV.

8. Conclusions

This thesis has presented the theoretical background underlying a number of exact results in three modifications of the supersymmetric field theory $\mathcal{N} = 4$ SYM, namely $\mathcal{N} = 2^*$ SYM, the β-deformation of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 4$ SYM with a defect. These results are part of the larger endeavor to understand quantum field theories in general and the link between gauge theories and string theory through the holographic principle.

The concepts of supersymmetric localization, integrable spin-chains and the Quantum Spectral Curve were introduced and their applications in the four appended papers were summarized. The conclusions and some open questions of these gathered works are:

- In the defect $\mathcal{N} = 4$ SYM of Paper I, treelevel two-point functions of $SU(2)$-operators and general length-two scalar operators can be calculated by the use of integrability techniques of the Heisenberg spin-chain together with an integrable Matrix Product State (MPS), representing the defect in the spin-chain picture. These results built on earlier work on one-point functions in the same setting and could rephrase the two-point functions as an operator in the spin-chain description of the one-point function of the $SU(2)$-operator. The known determinant formula for one-point functions could thus be reused.

  The approach relies on the length being two of one of the involved operators but as the one-point function results have been found for the full scalar sector, a potential generalization could be to work out the corresponding spin-chain operation for $SO(6)$-operators.

- Paper II combined the defect $\mathcal{N} = 4$ SYM with the β-deformation and found a vacuum solution analogous to the one in the undeformed case. The β-deformed MPS describing the defect in this setup was, however, shown to not correspond to any known notion of an integrable boundary state and hence obstructing earlier paths to determinant formulas for the one-point functions. Nevertheless, the simplest spin-chain overlaps with the MPS were calculated and turned out to have an underlying linear scaling dependence on a parameter k of the defect, in surprising difference to a power law dependence in the undeformed case.

  This latter discrepancy would be interesting to investigate from a holographic perspective. A confirming string theory comparison with the k-scaling of the operator $\mathrm{tr}(Z \dots Z)$ in the undeformed theory was done in [81] and it is an open question if a similar observation can be made for the deformed setup.

- Paper III employed localization in $\mathcal{N} = 2^*$ SYM to probe the strong coupling regime. An earlier proposed ansatz for the eigenvalue density function of the vector multiplet scalar, accounting for the large number of phase transitions accumulating at strong coupling, was put on firmer ground through a connection to the dual string theory. The phase transitions were furthermore shown to occur at the third order in the strong coupling expansion.

  That observation implies that the phase transitions could be perturbatively accessed in the holographic dual. The string theory calculations have as of yet not been pushed that far (the first correction was computed in [41]) but could in principle be done.

- Lastly in this thesis, Paper IV adapted known applications of the QSC in $\mathcal{N} = 4$ SYM to the β-deformation, proposing algorithms for finding leading solutions and the perturbations around them. They were exemplified for a number of operators belonging to the Konishi multiplet in the undeformed theory for which the anomalous dimensions were computed with a *Mathematica*-implementation, confirming earlier results and adding several loop orders.

  Future avenues of research could be to apply the algorithms to the $L = 2$ BMN vacuum multiplet and to classify all Konishi submultiplets. Potential improvements of the implementation is also an important question as the growing expressions of twist factors often quickly become challenging.

These are but a few, though very concrete, directions to pursue in the large effort of finding exact results in supersymmetric field theories and the overarching aim to uncover the inner workings of quantum field theories.

9. Sammanfattning

Kvantfältteori är den mest precisa beskrivning av den fysikaliska verkligheten som mänskligheten har lyckats formulera. Kvantitativa förutsägelser har verifierats av experiment upp till tolv siffrors noggrannhet vilket är långt bortom alla andra forskningsresultat. Det är en fantastisk framgång och ett bevis på kvantfältteorins betydelse för vår förståelse av naturen.

Snarare än en enskild teori är kvantfältteori ett matematiskt ramverk med breda användningsområden, från kondenserade materiens fysik till de fundamentala elementärpartiklar som utgör de minsta beståndsdelarna i universum. Den samlande egenskapen är att dessa fysikaliska objekt modelleras som excitationer av en samling variabler kallade fält.

Den mest kända kvantfältteorin är standardmodellen som sammanfattar i ett fåtal ekvationer alla elementärpartiklar och deras interaktioner via tre av de fyra fundamentala krafterna: de starka och svaga kärnkrafterna och den elektromagnetiska växelverkan. Modellen har varit extremt framgångsrik och har verifierats i ett hissnande antal experiment i partikelacceleratorer världen över.

Trots standardmodellens alla triumfer så finns det ett antal problem. Å ena sidan har vi det stora konceptuella tillkortakommandet att den inte är komplett. Varken den fjärde fundamentala kraften gravitation eller en förklaring till mörk materia ingår, så den är bara giltig som approximation för situationer när dessa är negligerbara[1]. Faktum är att det ännu inte finns en tillfredsställande teori för kvantgravitation, dvs. en teori som både inkluderar de kvantmekaniska principerna som råder på mikroskopiska avstånd och den attraktiva kraften mellan massiva föremål. Den ledande kandidaten är strängteori, i vilken de fundamentala objekten utgörs av en-dimensionella strängar vars vibrationer (excitationer) ger upphov till de partiklar vi observerar.

Å andra sidan finns det stora praktiska utmaningar. Trots den mycket kompakta och eleganta beskrivningen av elementärpartiklarnas fysik är mätbara storheter i praktiken väldigt svåra att beräkna. I allmänhet får man ta till approximationer, i synnerhet störningsräkning i vilken alla interaktioner antas vara små störningar ovanpå en lösning för hur systemet beter sig utan dem. Interaktionsstyrkan beskrivs matematiskt genom ett antal kopplingskonstanter

[1] För alla hittills realiserbara experiment är effekterna av gravitation och mörk materia helt försumbara varför resultaten är samstämmiga med standardmodellens förutsägelser. Vid beräkningar av extrema system, så som svarta hål och the Big Bang, är standardmodellen dock inte längre giltig.

som i störningsräkningar antas vara små. Resultaten från sådana beräkningar är därför bara giltiga under vissa omständigheter.

Bristen på exakta resultat och insyn i dynamiken vid starka interaktioner är stora hål i vår förståelse för, inte bara partikelfysik, utan för kvantfältteori i allmänhet. För att få en tydligare bild av den verklighet som kvantfältteorier beskriver krävs nya beräkningstekniker och någon form av exakta resultat som kan vägleda den teoretiska forskningen.

Den supersymmetriska kvantfältteorin $\mathcal{N} = 4$ Super-Yang-Mills (SYM) erbjuder precis sådana möjligheter. Tack vare sin stora uppsättning symmetrier har ett antal nya matematiska metoder kunnat utvecklas och exakta resultat fastställas.

Symmetrier, här i en vid matematisk mening, är det grundläggande språk som används för att beskriva elementarpartiklar. De systematiseras genom vad som inom matematiken kallas gruppteori och partikelfysikteorier klassifieras därför utefter vilka symmetrigrupper de har.

$\mathcal{N} = 4$ SYM har en mycket stor symmetrigrupp som innefattar maximal supersymmetri och (super)konform symmetri. Den föregående är en hypotetisk symmetri mellan de två partikelfamiljerna bosoner och fermioner och den senare innebär att teorin ser likadan ut oavsett på vilken skala den betraktas.

Trots att $\mathcal{N} = 4$ SYM inte är en realistisk model för vad som har observerats i naturen — supersymmetri saknar experimentell verifikation och konformalitet utesluter partiklar med massa — så fungerar den genom sina många förenklande symmetrier som ett teoretiskt laboratorium där idéer och nya tekniker kan utvecklas och testas. Tre sådana idéer, varpå denna avhandling vilar, är den holografiska principen, supersymmetrisk lokalisering och integrabilitet.

Den förstnämnda är en föreslagen dualitet mellan vissa kvantfältteorier och strängteori i en särskild geometri. Med “dualitet” menas att dessa två tillsynes vitt skilda teorier i själva verket beskriver samma fysik. Detta är häpnadsväckande eftersom de två teorierna har olika antal dimensioner; man kan betrakta kvantfältteorin som att den lever på randen av strängteorins universum men ändå fångar alla fysikaliska processer i sin beskrivning. Detta är vad som avses med den holografiska principen. De två olika beskrivningarna kompletterar varandra i och med att starka interaktioner i den ena motsvaras av svaga i den andra. Därmed kan man använda störningsräkning på ena sidan av dualiteten för att beräkna storheter vid starka interaktioner på den andra.

Den andra viktiga idéen, supersymmetrisk lokalisering, utnyttjar supersymmetri till att förenkla partitionsfunktionen, en funktion som används för att beräkna väntevärden av fysikaliska storheter. Vanligtvis innefattar den en integral över alla möjliga fältkonfigurationer men som via lokalisering kan begränsas till en delmängd av dem, ofta en offantlig förenkling då integrationsområdet kan reduceras från ett oändligt-dimensionellt rum till ett ändligt-dimensionellt. Det ger möjlighet att exakt beräkna ett antal (supersymmetriska) kvantiteter för godtyckliga värden på kopplingskonstanterna och kan således sprida

ljus över parameterområden som inte är tillgängliga genom traditionell störningsräkning eller den holografiska dualiteten.

Det sistnämnda genombrottet i $\mathcal{N} = 4$ SYM är förekomsten av integrabilitet. Att en model är kvantmekaniskt integrabel innebär löst att den besitter ett oändligt antal symmetrigeneratorer (som alla kommuterar) vilket kan användas för att i princip beräkna alla dess fysikaliska storheter. $\mathcal{N} = 4$ SYM upptäckes inneha en sådan integrabel struktur via en spinkedje-analogi där operatorers skaldimension kan beräknas på samma sätt som energinivåerna i ett en-dimensionellt system av partiklar med spin. Tekniskt innebär det att dilatationsoperatorn, i gränsen då gaugegruppens rank N går mot oändligheten (stora N-gränsen), verkar på komposita operatorer med ett enskilt matrisspår såsom Hamiltonianen på en isotropisk spinkedja där varje lokalt Hilbert rum är en representation av symmetrigruppen för fältteorin. I fallet för $\mathcal{N} = 4$ SYM är detta gruppen $PSU(2,2|4)$, eller eventuellt en undergrupp beroende på operatorn. Upptäckten av integrabilitet i $\mathcal{N} = 4$ SYM innebär att spektrumet av skaldimensioner, vilket är ytterst centralt för en konform fältteori, kan beräknas med kraftfulla spinkedje-tekniker kända och utvecklade under åtta decenier.

Dessa metoder har möjliggjort en rad nya resultat i den unikt symmetriska $\mathcal{N} = 4$ SYM. Målsättningen är emellertid att förstå kvantfältteorier i allmänhet så en viktig fråga är om, och i så fall hur, dessa lärdomar kan utökas till mer generella modeller.

Ett tillvägagångssätt är att starta ifrån $\mathcal{N} = 4$ SYM och sedan modifiera teorin på ett kontrollerat sätt så att de utformade beräkningsteknikerna fortfarande är applicerbara. Denna avhandling summerar fyra artiklar med resultat för tre sådana modifikationer:

i) **β-deformationen av $\mathcal{N} = 4$ SYM** vilken deformerar och bryter en inre symmetri (R-symmerin $SU(4)$) men bibehåller konformalitet och ($\mathcal{N} = 1$) supersymmetri,

ii) **den massiva deformationen $\mathcal{N} = 2^*$ SYM** i vilken ett antal partiklar ges massa ($\mathcal{N} = 2$ hypermultipleten med två komplexa skalärer och två fermioner). Det är således inte en konform teori då massan introducerar en energiskala men har fortfarande ($\mathcal{N} = 2$) supersymmetri.

iii) **$\mathcal{N} = 4$ SYM med en defekt**, dvs. ett tre-dimensionellt objekt som placeras som en rand och delar rumtiden i två områden. Det bryter den fyrdimensionella konforma symmetrin och möjliggör nya fysikaliska storheter (såsom en-punktsfuntioner och två-punktsfunktioner mellan operatorer med olika skaldimensioner).

Artikel I studerade ett specialfall av två-punktsfunktioner i $\mathcal{N} = 4$ SYM med en defekt. Trädgrafen för en operator i den så kallade $SU(2)$-sektorn och en operator bestående av två komplexa skalärer kunde tolkas som en spinkedjeoperation och återföras på kända resultat för en-punktsfunktioner.

I artikel II kombinerades den defekta $\mathcal{N} = 4$ SYM med β-deformationen och en vacuumlösning kunde hittas som gav de första resultaten för en-punktsfunktioner i denna sättning.

Artikel III använde supersymmetrisk lokalisering för att studera fasövergångar i $\mathcal{N} = 2^*$ SYM vid stark kopplingskonstant. Det kunde fastslås att dessa sker i nästnäst ledande ordningen varför motsvarande fasövergångar ännu inte har observerats i $\mathcal{N} = 2^*$ SYMs duala teori.

I den sista artkeln, artikel IV, anpassades den ledande integrabilitetsbaserade metoden för skaldimensionsberäkningar, den så kallade Quantum Spectral Curve, till konkreta beräkningar i β-deformationen av $\mathcal{N} = 4$ SYM. Algoritmer för att finna ledande lösningar och sedan utöka dem genom störningsräkning presenterades och applicerades på ett antal skaldimensioner, inklusive resultat upp till åttonde loop-ordningen.

Vi är ännu långt ifrån en fullständig förståelse för kvantfältteorier och en dag då förutsäglser för partikelexperiment kan göras med lätthet. Dessa exakta resultat och utforskandet av de tekniker som möjliggör dem är dock ett steg på vägen i denna gradvisa strävan.

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