Higgs Boson Decay to Two Photons

Probing the Standard Model

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Abstract

In July 2012, a particle in the mass region of 125 GeV was discovered at CERN's Large Hadron Collider (LHC). As predicted by the Standard Model, this particle would be required by the theory to maintain gauge symmetry and invariance by introducing proper mass eigenstates; this particle is of course the Higgs boson. Much work has been done, in theory and in experiment, to understand this particle and how it interacts.

In this paper, we look at how the Higgs particle fits into the standard model, and more specifically, we look at the important di-photon decay of the Higgs boson. This particular decay channel is analyzed and compared to known results. We then probe higher order corrections to the theory and try to understand how future calculations may take shape.

Contents

Introduction

The Standard Model of Particle Physics has been extremely successful and accurate in describing and theorizing interactions of elementary particles. Owing to the high energy capabilities of the LHC, the missing piece of the puzzle was found. The existence of the Higgs Boson had been theorized since the 1960's with calculations of its behavior in the SM well documented. Although the SM is incomplete, (i.e baryon symmetry, dark matter, neutrino mass, quantum gravity, etc.) it has done an extraordinary job in predicting most interactions.

In this paper, our main goal is to understand the specific process of Higgs decay width to two photons $\Gamma(h \to \gamma\gamma)$. With this in mind, we look at the SM as a whole; its symmetries and content. Since the standard model is a broken gauge theory, it is important to understand what this means and what causes this. We then detail the Higgs mechanism and its physical properties while looking at its production mechanism(s) at the LHC, along with its main decay channels.

The importance of $h \to \gamma\gamma$ lies in the clarity of its final state and its Higgs mass signature; the final states of the photons can be measured quite precisely. Even so, this channel is one of the least likely, with a branching ratio of around 0.2%; as we will see this is due to coupling strengths of the Higgs boson. It is well documented in the literature that the total decay width of the Higgs is calculated to be around 4 MeV. This is incredibly narrow for a particle of this size, but this is to be expected since these measurements scale with the Higgs coupled masses. Since the Higgs does not couple directly to photons, these processes are calculated at the one-loop level. In this paper, we will detail the Higgs di-photon decay channel and compare our findings with the literature. In doing so, we will also ask questions about future research and experimental avenues.

1 Gauge Theories in the Standard Model

The Standard Model is a spontaneously broken, non-abelian gauge theory that contains elementary particles (the language may change, but for the sake of sanity in this paper particles and fields will interchange equally). The corresponding field theory is a combination of free and interacting Lagrangians, which themselves are built by gauge symmetries. These gauge symmetries describe local transformations of the Lagrangian that leave it invariant; this is needed for a renormalizable Lagrangian, and in turn an consistent field theory. In this section we lay out the fundamentals of gauge symmetries for the abelian case, then the non-abelian gauge field theory becomes readily available with some slight modifications.

1.1 Abelian Gauge Field Theory

The first case we will look at is the simple Quantum Electrodynamics (QED) Lagrangian; this will help illustrate the gauge structure and dynamics of a system. We write the QED Lagrangian as,

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \not{D}_{\mu})\psi - m\bar{\psi}\psi, \tag{1}$$

where we use the notation $\not{D} \equiv \gamma^{\mu} D_{\mu}$ and our field strength tensor is given in terms of the gauge field,

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \tag{2}$$

Keep in mind also that throughout this paper we will drop spacetime dependence; however, this is only for brevity, all fields can be written as $\psi(x)$, for example. ψ can then be described as taking the form of the electron or positron fields, we can write that as:

$$\psi(x) = \begin{cases} \xi(x) & q_\psi = -1 \\ \eta(x) & q_\psi = +1 \end{cases}$$

As required by gauge invariance, the covariant derivative is defined by

$$D_\mu \psi = (\partial_\mu + ieq_\psi A_\mu)\psi(x) \tag{4}$$

where q_ψ is a placeholder for the corresponding fields charge. By construction, the QED Lagrangian is invariant under U(1) local gauge transformations which take the form

$$\begin{aligned} \psi &\to \exp[ieq_\psi \Lambda(x)]\psi(x) \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu \Lambda(x). \end{aligned} \tag{5}$$

Understand that the transformation of the field ψ can carry a negative sign in the exponent if the charge is opposite of the example above. This U(1) transformation can be applied to any field in the theory; for example, a complex scalar field $\Phi(x)$, as will be included later. Symmetries corresponding to U(1) transformations are quite trivial but this very simple example can be extended to understand more complex Lie groups and algebras.

1.2 Non-Abelian Gauge Theory

We can generalize the previous section by replacing the specific U(1) gauge group with some general non-abelian gauge group; we will call this G. This symmetry group, G, contains a set of Lie groups along with U(1). The ones of most interest for our purposes include, SU(n), SO(n), and U(1) (there are some others including Sp(n), but we wont need this here). We will now take a general field, this can be any field in the theory, and we will denote it as $\phi_i(x)$. Our general gauge transformation is then given by

$$\phi_i(x) \to U_i^j(g)\phi_j(x). \tag{6}$$

Here the indices are dimensional representations and g is an element that exists in G. We can then define our transformation matrix *locally* as

$$U(g(x)) = \exp[-ig\Lambda^a(x)T^a]. \tag{7}$$

Here T^a are generators of the theory (Lie Group) and g are our gauge couplings (in QED, 'e'). It is important to make the distinction between local transformations and global; local transformations will depend on spacetime coordinates. Linearly independent generators determine the dimension of the Lie group that satisfy the commutator

$$[T^a, T^b] = if^{abc}T^c. \tag{8}$$

Writing out the matrix elements in adjoint representation, the generators are then connected to the structure constants by

$$(T^a)_{bc} = -if_{abc} \tag{9}$$

From here it will be useful to understand the infinitesimal version of the group element from Eq. (7) just as we have with our matter and gauge fields. This is simply given by expansion of the exponent as

$$U(g(x)) \approx \mathbb{I} - ig\Lambda^a(x)T^a, \tag{10}$$

We can now follow a similar recipe to build a non-abelian gauge field theory. Following Eq. (6) and Eq. (10), we can reconstruct the group elements through infinitesimal gauge transformations. This transformation is given by

$$\begin{aligned} \phi_i(x) &\to \phi_i(x) + \delta\phi_i(x) \\ \delta\phi_i(x) &= -ig\Lambda^a(x)(T^a)_i^j\phi_j(x), \end{aligned} \tag{11}$$

this behavior will be the focus of building the theory. Our goal in this section is to understand the Yang-Mills Lagrangian and the consequences of the non-abelian structure. We begin by displaying the Lagrangian so that we can unpack it piece by piece:

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D})\psi - m\bar{\psi}\psi. \tag{12}$$

The extra index in our field strength is a consequence of our non-abelian gauge symmetry. We now want to turn our focus to the gauge field $A_\mu(x)$.

Since we want to describe describe transformations, it is useful to use the matrix-valued representation of our gauge field as

$$A_\mu(x) \equiv g A^a_\mu(x) T^a. \tag{13}$$

We want to do something similar with the covariant derivative. For a matter field that transforms according to the symmetry group G, we have a matrix representation of the covariant derivative given by

$$(D^\mu)^j_i = \delta^j_i \partial^\mu + ig A^a_\mu(x)(T^a)^j_i. \tag{14}$$

Looking at Eq. (5), the gauge field transforms as

$$A_\mu \to U A_\mu U^{-1} - iU(\partial_\mu U^{-1}) \tag{15}$$

where this satisfies our condition on the covariant derivative

$$D_\mu \Phi \equiv (\partial_\mu + iA_\mu)\Phi \to U D_\mu \Phi. \tag{16}$$

this equation and Eq. (9) allows us to write down our transformation law for the gauge field; namely

$$\begin{aligned} A^a_\mu &\to A^a_\mu + \delta A^a_\mu(x) \\ \delta A^a_\mu(x) &= g f^{abc} \Lambda^b A^c_\mu + \partial_\mu \Lambda^a. \end{aligned} \tag{17}$$

One then defines the adjoint representation of the covariant derivative as

$$D^{ab}_\mu \equiv \delta^{ab} \partial_\mu + g_a f^{abc} A^c_\mu. \tag{18}$$

We now have all the non-abelian machinery to understand the field strength tensor given previously in the Yang-Mills Lagrangian in Eq. (11). It will be quite visible when looking at the gauge invariant kinetic energy term, where the consequence of the non-abelian nature occurs. Remember that the commutator of our covariant derivatives is defined by

$$[D_\mu, D_\nu]\Phi = i\{\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\}\Phi. \tag{19}$$

In the QED case, the field strength tensor does not include this extra gauge field commutator (based on its simple group symmetry). We now have a dependence and relationship on these classes of generators, and thus we have a redefinition of the field strength tensor with the replacement A_μ^a and D_μ^{ab},

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c. \tag{20}$$

The nature of our non-abelian structure gives more complexity to the transformation of this field strength. The extra term becomes extremely important in our analysis, as it generates three and four-point functions in the theory (these being self interactions of the gauge fields). Here we have focused on the gauge term in the Lagrangian; however, it is then simple to construct any matter dependent Lagrangian term. Simply by replacing $\partial_\mu \to D_\mu$ and then requiring the appropriate matrix representation (generator) for each matter field. These matter fields become of utmost importance when looking at broken symmetries in the theory; this we will see in detail with the inclusion of the Higgs mechanism.

2 Feynman Rules

Constructing the Feyman Rules for our gauge theory is quite straight forward. In most cases, the usual Feynman rules allow one to take contractions at different orders of perturbation theory (S-matrix expansion). For our purposes, we will look mainly into functional methods as it becomes most instructive in identifying some 'outcast' terms.

When writing down these Feynman rules, one must write down the propagators of the theory; here, the issue arrises when looking at the gauge field propagator. If we look at the abelian case, we can solve the corresponding Green's function to arrive at a term $(g^{\mu\nu}\Box - \partial^{\mu}\partial^{\nu})\partial_{\mu} = 0$. This terms carries the implication that the the object inside the parentheses is not invertible (as one would do to solve for the gauge propagator; these create singularities in our functional integral). The solution(s) to this problem are Faddeev-Popov ghost fields detailed below. The general idea is to constrain the gauge by fixing it directly. We define our functional integral as

$$\int \mathcal{D}\mathcal{A} \exp iS[A^a_\mu] \tag{21}$$

where

$$S = \int d^4x[-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a] \tag{22}$$

is our action integral of the gauge field. The derivation follows, as is well documented from Peskin and Schroeder [1]. Since this integral is invariant under all local gauge transformations, we introduce a parameter such that $F(A^a_\mu) = 0$; this is simply our gauge fixing condition and is easily established by introducing a functional delta function (note that this term has no relation to our field strength tensor $F^a_{\mu\nu}$; F is just some chosen function). For this to be mathematically accurate, we also require that this term, when inserted into the functional integral, be equal to one. This term takes the form

$$1 = \int \mathcal{D}\Lambda(x)\delta(F(A^a_\mu))\det\left(\frac{\delta(F(A^a_\mu))}{\delta\Lambda}\right), \tag{23}$$

where here it is important to understand that we have gauged transformed our field A^a_μ according to Eq. (17), i.e

$$A^a_\mu(x) = A^a_\mu + gf^{abc}\Lambda^b A^c_\mu + \partial_\mu \Lambda^a. \tag{24}$$

This can also be written, and is sometimes regarded in the literature, by its adjoint representation

$$\begin{aligned}\left(A^\Lambda_\mu\right)^a &= A^a_\mu + \partial_\mu\Lambda^a + gf^{abc}\Lambda^b A^c_\mu \\ &= A^a_\mu + \frac{1}{g}\left(D_\mu\Lambda\right)^a.\end{aligned} \tag{25}$$

We can insert Eq. (23) into Eq. (21), and due to gauge invariance, change the order of integral such that our integral takes the form

$$\int \mathcal{D}\mathcal{A}e^{iS[A^a_\mu]} =) \int \mathcal{D}\Lambda \int \mathcal{D}\mathcal{A}\, e^{iS[A^a_\mu]}\delta(F(A^a_\mu))\mathrm{det}\left(\frac{\delta(F(A^a_\mu))}{\delta\Lambda}\right). \tag{26}$$

We now have an integral over A that is constrained by the delta function to physically relevant field configurations. To specify our gauge fixing function, we can work in the Lorentz gauge and set

$$F(A^a_\mu) = \partial^\mu A^a_\mu(x) - \omega^a(x), \tag{27}$$

where $\omega^a(x)$ can be treated as a scalar for our purposes. We then evaluate the determinant term as $\det\left(\frac{\delta(F(A^a_\mu))}{\delta\Lambda}\right) = \det(\frac{1}{g}\partial_\mu D^\mu)$. Inserting this and Eq. (25) into our integral gives us

$$\int \mathcal{D}\mathcal{A}e^{iS[A]} = \det\left(\frac{1}{g}\partial_\mu D^\mu\right)\int \mathcal{D}\Lambda \int \mathcal{D}\mathcal{A}\, e^{iS[A]}\delta\big(\partial^\mu A^a_\mu(x) - \omega(x)\big). \tag{28}$$

To manipulate this integral one step further, a Gaussian weighting function is introduced. We then integrate over all over all $\omega^a(x)$; one can see the delta function will 'pick out' our gauge factor

$$\begin{aligned} &= N(\xi)\int \mathcal{D}\omega \exp\left[-i\int d^4x\frac{(\omega^a)^2}{2\xi}\right]\det\left(\frac{1}{g}\partial_\mu D^\mu\right)\int \mathcal{D}\Lambda\times \\ &\qquad \int \mathcal{D}\mathcal{A}\, e^{iS[A]}\delta(\partial^\mu A^a_\mu(x) - \omega^a(x)) \\ &= N(\xi)\det\left(\frac{1}{g}\partial_\mu D^\mu\right)\int \mathcal{D}\Lambda\int \mathcal{D}\mathcal{A}\, e^{iS[A]}\exp\left[-i\int d^4x\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right].\end{aligned} \tag{29}$$

In our case we can just as easily drop the term $N(\xi)$ as it serves as some normalization constant. All all we have really done with this is added a term proportional to the derivative of our gauge field, essentially

$$\int \mathcal{D}\mathcal{A}\exp\left[i\int d^4x[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right], \tag{30}$$

and our Green function can now be solved quite easily with our new term ξ fixing our issue of singularity. We also can quickly deal with our term in the determinant; this is identified as our Faddeev-Popov ghost fields. These fields are well documented and given by anti-commuting fields; namely,

$$\det\left(\frac{1}{g}\partial_\mu D^\mu\right) = \int \mathcal{D}c\mathcal{D}\bar{c}\exp\left[i\int d^4x\,\bar{c}(-\partial_\mu D^\mu)c\right]. \tag{31}$$

Now that we have identified all fields, we can rewrite our functional integral given by Eq. (21) and Eq. (28) by including our ghost fields,

$$\begin{aligned} &\int \mathcal{D}\mathcal{A}\exp\left[i\int d^4(x)\big(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\big)\right] = \\ &= \int \mathcal{D}\mathcal{A}_\mu\mathcal{D}c\mathcal{D}\bar{c}\exp\left[i\int d^4x[\mathcal{L}_{YM} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}(-\partial_\mu D^\mu)c\right]. \end{aligned} \tag{32}$$

Here, we have abbreviated our original Yang-Mills Lagrangian; this is simply the non-abelian case of Eq. (1).

It is useful now to understand the formulation of further functional calculations. One could use the usual Feynman rules and Green's functions to solve for propagators, but here we outline the differences between Green's functions and the path integral formulation. If we begin by just looking at the gauge portion of our Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \tag{33}$$

we can manipulate this equation by integrating by parts to write this in a more useful way; namely

$$\mathcal{L} = \frac{1}{2}A^\mu_a\big(g_{\mu\nu}\,\Box + (\frac{1}{\xi} - 1)\partial_\mu\partial_\nu\big)A^\nu_a + \mathcal{O}(A). \tag{34}$$

Note here we have used the notation $\Box \equiv \partial^\mu\partial_\mu$ do identify the d'Alembert operator. The Green's function is then quite simply defined as

$$LG(x-x^{'}) = \delta(x-x^{'}) \tag{35}$$

where L is a linear differential operator and G is the corresponding propagator; this is why we write the Lagrangian as we did above; we can easily identify the operator portion of the Lagrangian. Our Green's function for our system is then straightforward and given by

$$\left(g_{\mu\nu}\,\Box + (\frac{1}{\xi}-1)\partial_\mu\partial_\nu\right)D_F^{\nu\lambda}(x-y) = \delta^{(4)}(x-y)\delta_\mu^\lambda. \tag{36}$$

Simply inverting this equation and Fourier transforming to momentum space we arrive at our propagator for our gauge field

$$iD_F^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \tag{37}$$

The prescription $i\epsilon$ is needed for convergence of our contour path integral. This term fixes our boundary conditions above (or below) the complex plane and tells us exactly how we integrate in the complex plane, moving the poles from a non-unique axis dependence; this gives us a unique solution and corrects singularities.

As we can see now, our propagator depends on our choice of gauge, ξ, in this paper we will regularly use the Feynman gauge: $\xi = 1$. We compare this now to the result that we obtain when using the path integral formulation. We have shown the basis of our functional integral from Eq. (21), now we want to understand functional methods. To do so, we need to define functional derivatives and the generating functional. We first define the generating functional for some field $\phi(x)$ as

$$Z[J] \equiv \int \mathcal{D}\phi \exp\left[i\int d^4x\left(\mathcal{L}+J(x)\phi(x)\right)\right]. \tag{38}$$

Here there has been an added source term $J(x)\phi(x)$. This additional term makes it incredibly easy to now derive correlation functions by taking functional derivatives. These derivatives have some special properties and they are given as follows:

$$\frac{\delta}{\delta J(x)}J(y) = \delta^{(4)}(x-y) \quad \text{and} \quad \frac{\delta}{\delta J(x)}\int d^4y\, J(y)\phi(y) = \phi(x). \tag{39}$$

We also have more complex derivatives that require chain rule manipulations, such as:

$$\frac{\delta}{\delta J(x)}\exp\left[i\int d^4y\, J(y)\phi(y)\right] = i\phi(x)\exp\left[i\int d^4y\, J(y)\phi(x)\right]. \tag{40}$$

If we then have derivative dependence in our integral, we can simply integrate by parts

$$\frac{\delta}{\delta J(x)}\int d^4(y)\,\partial_\mu J(y)V^\mu(y) = -\partial_\mu V^\mu(x). \tag{41}$$

We now have all the machinery to define and understand our two point (and higher) functions. Notice that each functional derivative brings down the term ϕ, these field contractions will indeed give us the propagators we are after. Our two point function is then defined as

$$\langle\Omega|\mathcal{T}\big(\phi(x)\phi(y)\big)|\Omega\rangle = \frac{1}{Z_0}\bigg(-i\frac{\delta}{\delta J(x)}\bigg)\bigg(-i\frac{\delta}{\delta J(y)}\bigg)Z[J]|_{J=0}. \tag{42}$$

In this definition, $Z_0 = Z[J = 0]$ and our derivatives are functions of the source terms. One can easily take higher order correlation functions by taking more derivatives; this is easily generalized as we will show. To understand how we can easily derive correlation functions, we will explicitly show a simple example then use the same technique for our non-abelian gauge fields.

If we take $\phi(x)$ to be the Klein Gordon field, we can write our generating functional as

$$\int d^4(x)\,[\mathcal{L} + J\phi] = \int d^4(x)[\frac{1}{2}\phi(-\partial - m^2 + i\epsilon)\phi + J\phi]. \tag{43}$$

Note that we have dropped the index and spacetime dependence for brevity. Now what we can do is make a clever shift in our field by identifying the corresponding Green's function for the field operator, i.e the propagator for the scalar field $\phi(x)$ (these are the same steps taken to arrive at Eq. (34)). the field is now given by

$$\phi(x)' = \phi(x) - i\int d^4(y)D_F(x-y)J(y), \tag{44}$$

inserting this change of variables into Eq. (42), we arrive at a new generating functional exponential term of

$$\begin{aligned}\int d^4(x)[\mathcal{L}+J\phi] =& \\ &= \int d^4(x)[\frac{1}{2}\phi'(-\partial^2-m^2+i\epsilon)\phi'] \\ &\quad - \int d^4xd^4y\frac{1}{2}J(x)(-iD_F(x-y)J(y).\end{aligned} \tag{45}$$

In terms of our generating functional, this is exactly what we want. we have a term that is our original Lagrangian but now dependent on ϕ', and then we have a term that is only source terms with a propagator. to write this explicitly, this is given as

$$Z[J] = \int \mathcal{D}\phi' \exp\Big[i\int d^4x\mathcal{L}_0(\phi')\Big]\exp\Big[-i\int d^4xd^4y\frac{1}{2}J(x)(-iD_F(x-y)J(y)\Big]. \tag{46}$$

The term dependent on ϕ' is simply Z_0 and will be cancelled out when looking at Eq. (42). From this point we can foreshadow our answer just by looking at this equation and what we now know about functional derivatives. For the sake of completeness, we take our two point function of the Klein Gordon field, i.e

$$\begin{aligned}&\langle\Omega|\mathcal{T}\big(\phi(x)\phi(y)\big)|\Omega\rangle = \\ &= -\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\exp\Big[-i\int d^4xd^4y\frac{1}{2}J(x)(-iD_F(x-y)J(y)\Big]\Big|_{J=0} \\ &= -\frac{\delta}{\delta J(x_1)}\Big[-\frac{1}{2}\int d^4yD_F(x_2-y)J(y)-\frac{1}{2}\int d^4xJ(x)D_F(x-x_2)\Big]\frac{Z[J]}{Z_0}\Big|_{J=0} \\ &= D_F(x_1-x_2).\end{aligned} \tag{47}$$

This method indeed works for any higher point function as well. Using the same methods, one can show quite easily that

$$\begin{aligned}&\langle\Omega|\mathcal{T}\big(\phi_1\phi_2\phi_3\phi_4\big)|\Omega\rangle = \\ &= D_F(3-4)D_F(1-2)+D_F(2-4)D_F(1-3)+D_F(1-4)D_F(2-3),\end{aligned} \tag{48}$$

this is a combination of all possible contractions, just as is the case if one were to use the S-matrix formalism and contract fields. The formalism of the derivation above will be precisely how we arrive at our non-abelian gauge field propagator. We can now use these methods to solve the Lagrangian we have introduced

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial^\mu \bar{c}^a)D^{ac}_\mu c^c. \tag{49}$$

If we look at each term in the same way we have done in the derivation above (for the Klein Gordon field), we can easily deduce the propagators for the theory. One finds the propagators given by

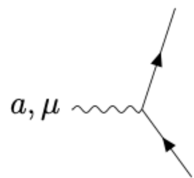
$$\begin{aligned} D^{ab}_G(k) &= \frac{i\delta^{ab}}{k^2 + i\epsilon} \\ D^{\mu\nu}_F(k) &= \frac{-i}{k^2 + i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \end{aligned} \tag{50}$$

Here we recover the same propagator for our gauge field as we did when solving the corresponding Greens function. These are both related to the propagator in configuration space by a simple Fourier transformation; for our purposes, it will be more useful to write these propagators in momentum space. The functional approach to deriving propagators is somewhat straightforward and instructive; however, to understand the interaction terms that produce vertex contributions, we must expand our Lagrangian. We understand this by looking at our Yang-Mills Lagrangian; writing it as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$, where the interaction term is just a power series expansion in g. This expansion is taken to non-linear terms with a matter field given by ψ;

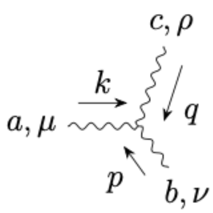
$$\mathcal{L} = \mathcal{L}_0 + gA^a_\lambda \bar{\psi}\gamma^\lambda T^a \psi - gf^{abc}(\partial_\kappa A^a_\lambda)A^{\kappa b}A^{\lambda c} - \frac{1}{4}g^2(f^{eab}A^a_\kappa A^b_\lambda)(f^{ecd}A^{\kappa c}A^{\lambda d}). \tag{51}$$

Here one can just read off vertex contributions while understanding that derivatives bring down momentum dependence. It is instructive to understand this with the use of Feynman Diagrams (see Fig. 1). The first diagram in the figure is read from the first non-linear term in the expansion; one can clearly see how the symmetry groups play a role at this level. The ideas proposed here will be key in understanding the Standard Model Lagrangian and its particle content.

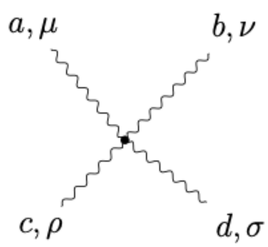
Figure 1: Feynman Rules for Yang-Mills Gauge Interactions



$$= ig\gamma^{\mu}T^{a}$$



$$= gf^{abc}[g^{\mu\nu}(k-p)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\rho\mu}(q-k)^{\nu}]$$



$$\begin{aligned} = -ig^{2}[&f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})] \end{aligned}$$

3 The Standard Model

The Standard Model of Particle Physics encompasses all interactions of elementary particles, and although it is an intimidating equation, it can be well understood using gauge theory and symmetries. We want to be able to describe strong, weak, and electromagnetic interactions that take place between the particles described in Table 1. To do so, we must outline each theory contained in the SM; namely the strong nuclear force, the weak nuclear force, and the electromagnetic force.

3.1 Gauge Theory and Particle Content of the Standard Model

To build up the gauge theory of the Standard Model, we can start by introducing the notation G_{SM} to encompass our gauge group. The theory is self contained by a combination of Quantum Chromodynamics and Electroweak Theory.

3.1.1 Quantum Chromodynamics (QCD)

QCD is mediated by massless gluon fields interacting with themselves and quarks with the free field Lagrangian for a single quark written, naively, as

$$\mathcal{L}_0 = \sum_{i=1}^{3} \bar{q}_i (i\not{\partial} - m_q) q_i. \tag{52}$$

Quarks themselves have different colors: red, blue, green; summed over here) and a corresponding anti-particle. The theory is governed by the gauge group

SU(3) with transformations obeying

$$q_i \rightarrow q_i^{'} = U_{ij} q_j, \tag{53}$$

with a transformation matrix defined by

$$U(\epsilon_a) = \exp\left(-i \sum_{a=1}^{8} \epsilon_a \frac{\lambda_a}{2}\right). \tag{54}$$

Here we have emphasized the sum over $N^2 - 1$ parameters (as this will be different for each gauge group). Note also that λ_a are traceless generators defined as the Gell-Mann matrices that obey the Lie algebra

$$[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c. \tag{55}$$

Two unique requirements of QCD are as follows: the three colors must be connected through gauge transformations, and quarks and anti-quarks must transform differently. These requirements tell us directly that are already set in an irreducible representation of a gauge group; thus, they are not some subgroup of SU(2). We can then define the covariant derivative as

$$(D^\mu q)_i = [(\partial^\mu + i g_s G_a^\mu \frac{\lambda_{a,ij}}{2})q]_i, \tag{56}$$

explicitly writing out the matrix indices. Knowing the construction of the field strength tensor, we can write out our QCD Lagrangian as

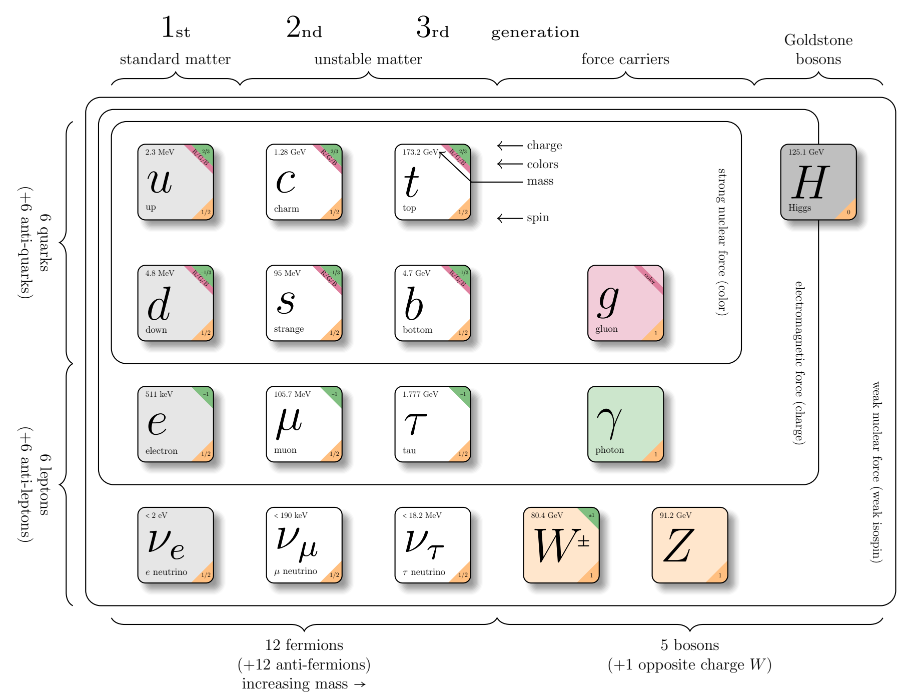
$$\mathcal{L}_{QCD} = -\frac{1}{2}(G_{\mu\nu} G^{\mu\nu}) + \sum_{f=1}^{N_f} \bar{q}_f (i \not{D} - m_{fc}) q_f. \tag{57}$$

We have also introduced our strong gauge coupling constant g_s that is some measure of the strength of quark-gluon interactions. The theory is consistent with our analysis of non-abelian gauge theories stated earlier. Soon, our covariant derivative will be self-consistent with G_{SM}; however, until we understand each piece we will build it in pieces.

3.1.2 Electroweak Theory

Since we are not incredibly interested in QCD in this paper, we have simply summarized the main points to understand the Lagrangian and its gauge group. However, it is of utmost importance that we detail electroweak theory as this will be the main driver of the physics we discuss. Even though we could deal with weak interactions separate from QED, it is instructive to combine them as we will need to in any case. The symmetry of the Standard Model is largely held up by electroweak interactions as we will come to see. To understand the intricacies of the theory we must first start by knowing how leptons are dealt with. With six leptons coming in families of three, there can only be a choice of SU(2) doublets or singlets.

Table 1: Particle Content of the Standard Model



Leptons are split into left-handed and right-handed components as follows (with ψ_i being a sum over leptons, for example):

$$\psi_{iL} = (1 - \gamma_5)\frac{\psi_i}{2} \quad , \quad \psi_{iR} = \frac{1}{2}(1 + \gamma_5)\nu_i(x). \tag{58}$$

The uniqueness of Electroweak theory comes from the fact that left-handed and right-handed components transform according to different gauge groups, and the theory is mediated by massive gauge bosons; $W^{\pm}_{\mu}$, Z^0_{μ}, and A_{μ}. As we now know, each one of these corresponds to a generator of their respective gauge group G. The only way in which one can construct a unique algebra with four generators is a crossed gauge group of $G = SU(2) \times U(1)$. We then need to construct a linear combination of the operational generators of the theory. We assign the generators of SU(2) as T_i where $i = 1, 2, 3$; remembering also that U(1) has a hypercharge described by Y. Then we can write the total electric charge operator, Q, as

$$Q = T_3 + \frac{1}{2}Y. \tag{59}$$

We are now able to more specifically identify the corresponding transformations. our left-handed fields transform according to SU(2)

$$\psi^i_L(x) \to e^{i\alpha^a(x)\tau^a}\psi^i_L(x), \tag{60}$$

where generators are identified as the Pauli matrices such that $\tau^a = \sigma^a/2$, and our right-handed fields transform under the $U(1)_Y$ gauge transformations as

$$\psi^i_R \to e^{i\beta/2}\psi^i_R. \tag{61}$$

This can be written, for completeness, as

$$G \propto SU(2)_L \times U(1)_Y = e^{i\alpha^a(x)\tau^a} e^{i\beta/2} \tag{62}$$

where the subscripts gives clarity to the corresponding field and classification. Notice also that our hypercharge shows up in our gauge transformation in the β term; when we introduce the Higgs field and its corresponding vacuum expectation value (henceforth "vev"), this transformation leaves the theory invariant (this puts an extra requirement on α^a).

We can now begin to build up the Lagrangian of the theory by understanding the gauge conditions and other constraints. Because of the different transformations of left and right-handed fields, parity and charge conjugation are broken. However, we have seen the combined transformation given by Eq. (62) leaves the theory invariant; this is CP symmetry. In this way $W^{\pm}$ bosons couple to left-handed doublets and right-handed antifermions. With this in mind we can start with a simple free Lagrangian for the theory; keeping our particle content in mind from Table 1, we have

$$\mathcal{L}_0 = i\bar{\psi}_j(x)\gamma^{\mu}\partial_{\mu}\psi_j(x) \tag{63}$$

As we have previously explained, to preserve invariance we must introduce a gauge dependent covariant derivative. We now have four gauge bosons in the theory, so we write this as

$$D_\mu \psi_i(x) = [\partial_\mu - igW^a_\mu(x)\tau^a - ig' B_\mu(x)]\psi_i(x); \tag{64}$$

now the Lagrangian above can be written as

$$\mathcal{L}_0 = i\bar{\psi}_j(x)\gamma^\mu D_\mu \psi_j(x). \tag{65}$$

We can then build up our gauge kinetic terms by first writing them explicitly as

$$\begin{aligned} B_{\mu\nu} &\equiv \partial_\mu B_\nu - \partial_\nu B_\mu \\ W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ijk} W^j_\mu W^k_\nu. \end{aligned} \tag{66}$$

An important note is that we can also write $\tilde{W}_{\mu\nu} \equiv \frac{\sigma_i}{2} W^i_{\mu\nu}$, as this is a common construction in the theory. Also, one can compare notation here to the general gauge theory described in Section 1 where we used gauge indices a, b, c; all we have done now is specifically describe SU(2) with gauge indices i, j, k (this is just by use of convention). Since we already have the recipe for constructing gauge kinetic terms, we just need to identify that $B_{\mu\nu}$ is invariant under our gauge transformation and $\tilde{W}_{\mu\nu}$ transforms in the covariant way just as our non-abelian gauge field A^a_μ. With this information, the kinetic terms of the Lagrangian are written as

$$\mathcal{L} \propto -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{\mu\nu}_i. \tag{67}$$

Now one may ask the question, how can you write a mass terms that couples to fields that transform differently? This question becomes an important one as left and right-handed fields would be 'mixed' in writing these mass terms; as we know this cannot be possible. This then would assume that the fields contained in the theory, as we have written it, are indeed massless (of course we know this to be untrue). This is something we will deal with when introducing the Higgs field; for now, it will suffice to understand the interactions of Electroweak theory without the details of mass and symmetry breaking.

3.2 Electroweak Interactions

In the process of understanding Higgs di-photon decay, we will detail calculations that describe interactions of Electroweak theory. With this in mind, we will take this section to describes the *types* of interactions without detailed calculation. These types of interactions are given as follows: charged-current interactions, neutral-current interactions, and gauge boson self-interactions.

3.2.1 Charged-Current Interactions

Remembering our CP symmetry and coupling constraints, we can write a piece of the Lagrangian as

$$\mathcal{L} \propto g\bar{\psi}_L(x)\gamma^\mu \tilde{W}_\mu \psi_L(x) + g^{'} B_\mu y_j \bar{\psi}_j(x)\gamma^\mu \psi_j(x), \tag{68}$$

where y_j are placeholders for the corresponding fermion hypercharges. To understand the more specific couplings it is important to unpack the matrix of $\tilde{W}_\mu$. To visualize this representation, we unpack the term in the following way

$$\tilde{W}_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}W_\mu^3 & W_\mu^+ \\ W_\mu^- & -\sqrt{2}W_\mu^3 \end{pmatrix} \tag{69}$$

where the linear combinations of the W boson as written in the matrix is given by

$$W_\mu^\pm = (W_\mu^1 \mp iW_\mu^2)/\sqrt{2}. \tag{70}$$

We can now write out an interaction term for our Lagrangian consisting of this vector boson coupled to lepton and quark fields. Using Eq. (58) and (69), the Lagrangian is written as

$$\mathcal{L} \propto \frac{g}{2\sqrt{2}}\Big[W_\mu^+\big(\bar{u}\gamma^\mu(1-\gamma_5)d + \bar{\nu}_e\gamma^\mu(1-\gamma_5)e\big)\Big] \tag{71}$$

Here we have distinguished quark and lepton fields specifically to show how the charged boson couples to the respective fields. This is an incomplete description as we cannot define mass here, thus interactions would become arbitrarily long range force producers.

3.2.2 Neutral-Current Interactions

Given that the photon couples and interacts in the same way with both left-handed and right-handed fermions, there must be some relationship between W_μ and B_μ, with the electromagnetic field. This relationship is given by a simple rotation matrix

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}, \tag{72}$$

and from this we can easily construct a Lagrangian by noticing the simple matrix construction change from Eq. (71):

$$\mathcal{L} \propto \bar{\psi}_j(x)\gamma^\mu \Big(A_\mu[g\tau^3 \sin\theta_W + g' y_j \cos\theta_W] + Z_\mu[g\tau^3 \cos\theta_W - g' y_j \sin\theta_W] \Big) \psi_j. \tag{73}$$

Here we see the structure produced by the generators τ while also noticing that we can impose the coupling conditions such that, $g \sin\theta_W = g' \cos\theta_W = e$, the electric charge.

3.2.3 Gauge Self-Interactions

These interactions are quite straight forward and come from the kinetic terms derived in Eq. (67). Expanding these terms, we find the Lagrangian contains cubic and quartic interactions (self). SU(2) algebra requires that a a pair of charged W bosons always couple to photons and Z bosons. If we expand the Lagrangian in Eq. (67) to third order, we can get terms that look like this:

$$\begin{aligned} \mathcal{L} \propto -ie\cot\theta_W [(\partial^\mu W^{\nu -} - \partial^\nu W^{\mu -}) W_\mu^+ Z_\nu - (\partial^\mu W^{\nu +} - \partial^\nu W^{\mu +}) W_\mu^- Z_\nu + \\ + W_\nu^- W_\nu^+ (\partial^\mu Z^\nu - \partial^\nu Z^\mu)]. \end{aligned} \tag{74}$$

Terms and interactions given here are common in electroweak theory, especially when taking into account interactions with photons; to visualize this, the Lagrangian above will reproduce a similar form but with the change

$Z_\mu \rightarrow A_\mu$.

As we have touched on briefly in this section, we need to extend our understanding further if we are to describe actual electroweak processes. We know through experiment (and of course, theory) that these physical gauge boson (excluding the photon field) must be massive to mediate these interactions. Since we have built up a Lagrangian through gauge symmetry without mass terms, there must be a mechanism that breaks this symmetry. This is what will be described in the next section and will allow us to understand and predict physical processes.

4 Spontaneous Symmetry Breaking

We now want to be able to describe and understand the Higgs mechanism and how it gives rise to mass in the theory. This is paramount to understanding physical interactions in the Standard Model and how the Higgs mediates mass in the theory. We will start with a very simple symmetry breaking model, in a discrete sense, then show how this works in a continuous model. We then will have the machinery to detail the Higgs Mechanism.

4.1 Discrete Case

The idea of symmetry breaking can be easily show in the case of ϕ^4 theory. Given the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2}\mu^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x), \tag{75}$$

where we have just defined μ as our mass scale term (given by the replacement $m^2 \rightarrow -\mu^2$), we can identify our potential as

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2(x) + \frac{\lambda}{4!}\phi^4(x). \tag{76}$$

We want to then minimize the potential, by taking the derivative with respect to ϕ, giving us the minimum-energy configuration of the classical field, or the vacuum expectation value (vev) of $\phi(x)$, noted as ϕ_0:

$$\phi_0 = \pm v = \pm\sqrt{\frac{6}{\lambda}}\mu. \tag{77}$$

The broken *discrete* symmetry is easily seen if we make a simple change of variables, e

$$\phi(x) = v + \sigma(x). \tag{78}$$

Here we classify our field by a small perturbation from our vev, given by σ It is then simple to rewrite our Lagrangian in terms of $\sigma(x)$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma(x))^2 - \frac{1}{2}(2\mu^2)\sigma^2(x) - \sqrt{\frac{\lambda}{6}}\mu\sigma^3(x) - \frac{\lambda}{4!}\sigma^4(x). \tag{79}$$

This is a scalar field of mass $\sqrt{2\mu}$ and due to our redefinition of the field in Eq. (78), any linear form of $\sigma(x)$ is non-physical and should vanish. We also now have $\sigma^3(x)$ and $\sigma^4(x)$ interactions; consequently our parity symmetry is lost. In a discrete and simple case this is spontaneously broken symmetry. We can also generalize this procedure for the continuous case.

4.2 Continuous Case

As we will use a very similar procedure as the discrete case, we will summarize the results and comment on the differences. We are now dealing with a set of $i = 1, 2, ..., N$ real scalar fields $\phi^i(x)$ such that our Lagrangian is now

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i(x))^2 + \frac{1}{2}\mu^2(\phi^i(x))^2 - \frac{\lambda}{4!}[(\phi^i(x))^2]^2. \tag{80}$$

The fields also are invariant under the transformations of the group *O(N)* (i.e N dimensional orthogonal group). Minimizing the potential, we arrive at the vacuum expectation value of

$$\phi_0^i = \sqrt{\frac{\mu^2}{\lambda}}. \tag{81}$$

We then shift the fields by introducing a similar term as before but also including a field π such that our fields are written as

$$\phi^i(x) = (\pi^k(x), v + \sigma(x)), \qquad k = 1, ..., N-1, \tag{82}$$

inserting this into our Lagrangian, we arrive at

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi_k(x))^2 + \frac{1}{2}(\partial_\mu \sigma)^2(x) - \frac{1}{2}(2\mu^2)\sigma^2(x) - \sqrt{\lambda}\mu\sigma(x)\big(\pi_k^2(x) - \sigma^2(x)\big) - \\ -\frac{\lambda}{4}\big(\pi_k^2(x) + \sigma^2(x)\big)^2. \tag{83}$$

Here we have introduced π fields that transform under *N-1* dimensional group operations of *O(N)*. We lose the *N*-dimensional symmetry; however, we do have *N-1* symmetry which describes rotations of the $\pi(x)$ fields. Just as in the discrete case above, $\sigma(x)$ represents a massive field whereas π is massless. This can be understood simply by inspection of the Lagrangian. More elegantly, we can see this by visualizing the potential from Eq. (80). This is famously know as the Mexican hat potential and is invariant under a global phase transformation, i.e $\exp[i\theta]$. What this tells us is there is an entire circle at the bottom of this well-potential where the field can be perturbed along with no requirement of energy. This is exactly the definition of a massless field and a more general definition of this result is defined by Goldstone's theorem.

4.3 Goldstone's Theorem

Goldstone's Theorem states for every spontaneously broken continuous symmetry, the theory contains a massless particle. If we are given a rotation in *N* dimensions, the rotation can be that of any *N(N-1)/2* planes. As we saw in the linear sigma model, after a spontaneous symmetry breaking, we are left with *(N-1)(N-2)/2* symmetries. We can read off the number of leftover symmetries is then just *N-1* (if, in fact, we are dealing with $N = 3$; Hence we have the massless field π in the previous example).

To show that this theorem is not only conceptually sound, but also mathematically so, we must show that the general symmetry of a given Lagrangian need not be a symmetry of ϕ_0 (the *vev*). This is done by minimizing the potential and identifying the mass eigenvalues evaluated at the *vev*. We can take a set of fields, call them $\phi^a(x)$, and write a very general Lagrangian

$$\mathcal{L} = (\text{kinetic terms}) - V(\phi). \tag{84}$$

We then choose a constant field ϕ_0^a that minimizes V. When we expand V about this minimum we arrive at our mass term in the form,

$$\left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}V\right)_{\phi_0} = m_{ab}^2. \tag{85}$$

This equality is a symmetric matrix whose eigenvalues give the masses of the fields. We want to show that any symmetry that is not also a symmetry of ϕ_0, must have a zero valued mass eigenvalue. If we now take a transformation

$$\phi^a \to \phi^a + \alpha\delta^a(\phi), \tag{86}$$

with δ^a just a placeholder function of all $\phi' s$, the potential now must be invariant under this transformation. This statement is the same as writing

$$V(\phi^a) = V(\phi^a + \alpha\delta^a(\phi)). \tag{87}$$

Differentiating with respect to ϕ and evaluating the field at the minimum, we get

$$0 = \left(\frac{\partial\delta^a}{\partial\phi^b}\right)_{\phi_0}\left(\frac{\partial V}{\partial\phi^a}\right)_{\phi_0} + \delta^a(\phi_0)\left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}V\right)_{\phi_0}. \tag{88}$$

The first term in this equation vanishes because ϕ_0 is a minimun of V. It is then trivial that our mass term must be zero to satisfy the equation. This is simply a general proof of what we have seen in the previous section. We can identify this proof more specifically by attributing $\delta^a(\phi_0)$ as a generator of the theory (in fact, it is this); for example, T^a from out gauge theory. This derivation corresponds to a broken generator such that $T^a(\phi) \neq 0$. This case will always correspond to a null mass eigenstate, and thus a massless field.

4.4 The Higgs Mechanism

We now have the machinery to understand the Higgs mechanism and its crucial role in the Standard Model. This can be shown by introducing a complex scalar field into our QED Lagrangian as follows

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu\phi|^2 - V(\phi). \tag{89}$$

It is important to note that this is indeed still a model that is a useful description of how the mechanism works in practice. Here, we also retain the usual $U(1)$ symmetry fo the theory. It is then instructive to use the Goldstone model to describe the potential $V(\phi)$ as

$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2. \tag{90}$$

As described earlier, a choice of $\mu > 0$ will spontaneously break our symmetry. When we do so, we will get a result identical to that of Eq. (81). However, the subtlety lies in the choice of parameterization of the complex field. The choice follows from that of (Peskin and Schroeder*) and is given by

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}\big(\phi_1(x) + i\phi_2(x)\big). \tag{91}$$

Taking into account the value of the *vev* as given by Eq. (81), the potential term can be rewritten such that the field ϕ_1 acquires mass and ϕ_2 is massless. At first glance this can be seen as just a consequence of choice; however, these fields are in fact Goldstone bosons and are paramount for mass eigenstates. Although they are unphysical, we have seen in the section before that they are required to account for the extra degrees of freedom.

We can do this same analysis with the kinetic energy term by expanding it in terms of our new complex field. Upon expansion we have

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \sqrt{2}e\phi_0 \cdot A_\mu\partial^\mu\phi_2 + e^2\phi_0^2A_\mu A^\mu + (\ldots), \tag{92}$$

where we have left out some higher order terms of the theory. What we want to identify here is a peculiar term, namely a new mass values for our photon field:

$$\Delta\mathcal{L} = \frac{1}{2}m_A^2 A_\mu A^\mu, \qquad \text{with} \quad m_A^2 = 2e^2\phi_0^2. \tag{93}$$

This is due to the non-vanishing *vev* and is only possible in the case where a massless scalar particle creates a pole in the vacuum polarization amplitude.

To see this more clearly, one can treat this mass term as a vertex in the theory just as one would do with any Feynman diagram. What one finds is a vacuum polarization amplitude that is properly transverse as required by the theory; structurally, it take the form

$$\text{Diagram} = im_A^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right). \tag{94}$$

This is exactly the mechanism in which massive vector bosons acquire their mass. It is quite popular in the literature to say that these boson 'eat' the Goldstone bosons to acquire this extra degree of freedom. This is due to the fact that one can choose a gauge such that this massless boson is eliminated from the theory. There is also the fact that massive gauge boson have three degrees of freedom while massless ones have two; these massive fields gain an 'extra' degree of freedom from this Goldstone boson. There are indeed more intricacies to the quantization of spontaneously broken field theory, but this description will help to understand the mechanism in a more detailed analysis.

5 The Higgs Boson

Until now, we have excluded mass from the theory, specifically when describing massive vector bosons. The way we introduce, is by the inclusion of the Higgs field. The importance of the Higgs cannot be understated; it is necessary to make sense of the Standard Model. Now that we have shown how the mechanism in which it is introduced produces mass terms, we now want to understand its specific behavior in the theory.

We start by parameterizing the Higgs *vev*, remembering our structure from Eq. (91), by

$$\langle \phi \rangle = \phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{95}$$

Since we are mainly concerned by the introduction of our mass terms, we will focus on how ϕ_0 couples in the theory (somewhat stepping away from an analysis of the Goldstone bosons; these will return later).

5.1 Fermion Mass

Although the Higgs mechanism is imperative to the theory, it has been theoretically hypothesized and necessary for decades. In order to be consistent with the theories built up for gauge bosons, this scalar field must be consistent in terms in gauge invariance and quantum number; the Higgs field is thus a spinor of SU(2) with hypercharge $Y = 1/2$. Before introducing the Higgs field, one could naively write a mass term for fermions as

$$\Delta \mathcal{L}_f = -m_e(\bar{e}_L e_R + \bar{e}_R e_L). \tag{96}$$

Here we label $\Delta \mathcal{L}_f$ to represent some piece of a Lagrangian that includes fermions. The two objects in the above mass coupling equation do not couple or transform in the same gauge representation. Understand that Eq. (96) is completely wrong; the structure of mass mixing does not maintain the symmetry of the theory. Now, with the Higgs field and the use of Eq. (95), we can introduce a dimensionless coupling constant and write it as

$$\Delta\mathcal{L}_f = -\frac{1}{\sqrt{2}}\lambda_e v \bar{e}_L e_R + \text{h.c.} \tag{97}$$

Where we have defined 'h.c' as the hermitian conjugate of the previous term. It is important to note that this coupling is a consequence of the the broken symmetry caused by ϕ picking up a nonzero *vev*. One could write a more general expression without breaking the symmetry (introduction of the *vev*) by writing this as

$$\Delta\mathcal{L}_f = -\lambda_e \bar{\Psi}_L \phi\, \psi_R + h.c. \tag{98}$$

Here we emphasize the the left-handed object is indeed a doublet in the theory that contracts with ϕ. The mass term for the electron is easily identifiable from Eq. (97), and given by

$$m_e = \frac{\lambda_e v}{\sqrt{2}}. \tag{99}$$

Mass terms for quarks are manifested in the same way with a little more care given to the initial construction of the Lagrangian. In this way, masses are all dependent on this *vev* value, and upon introduction of the actual Higgs field (in unitary gauge), $h(x)$, we see a structure that often takes the form

$$\mathcal{L}_f = -m_f \bar{\psi}_f \psi_f (1 + \frac{h}{v}). \tag{100}$$

The unitary gauge becomes useful when looking at the Higgs field simply due to its ease of use in the theory. This is defined as

$$\phi(x) = U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \tag{101}$$

where U(x) is a general unitary matrix transformation. and will also be used when showing the couplings in a full description of electroweak theory.

5.2 Gauge-Boson Mass

In Section 3b, we saw the structure and behavior of gauge bosons in electroweak theory. We now need to introduce the scalar field into the theory to identify their respective mass spectrums. Our Higgs field needs to obey the symmetry of the theory and thus transforms as

$$\phi \to e^{i\alpha^a\tau^a}e^{i\beta/2}\phi. \tag{102}$$

Here, $\tau^a = \sigma^a/2$ are the usual Pauli matrices. Notice here the term constituting $U(1)$ symmetry has been given a scalar field charge of $+1/2$. When our scalar field has a parametrized *vev* as in Eq. (95), then a gauge transformation with $\alpha^1 = \alpha^2 = 0$ and $\alpha^3 = \beta$ leaves ϕ_0 invariant. This is consistent with Goldstone's theorem due to this combination of generators producing one massless gauge boson and three massive ones.

To see the details of this mass spectrum, we define the covariant derivative acting on our scalar field as

$$D_\mu\phi = (\partial_\mu - igA^a_\mu\tau^a - \frac{i}{2}g^{'}B_\mu)\phi. \tag{103}$$

One must take care to denote proper coupling constants to the corresponding gauge groups due to commuting variables (distinction given here by 'prime'). To look at the form of these mass spectrums, we square the covariant derivative term and break the symmetry as

$$\mathcal{L}_{KE} = \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}(gA^a_\mu\tau^a + \frac{1}{2}g^{'}B_\mu)(gA^{b\mu}\tau^b + \frac{1}{2}g^{'}B^\mu)\begin{pmatrix}0\\ v\end{pmatrix}. \tag{104}$$

It is important to remember that our gauge bosons are related by matrix operations that can be understood as a change of basis. We can then evaluate this Lagrangian term by components, using Pauli matrices, given by

$$\mathcal{L}_{KE} = \frac{1}{2}\frac{v^2}{4}[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (-gA^3_\mu + g^{'}B_\mu)^2]. \tag{105}$$

Now it is possible to identify these gauge bosons in components as

$$
\begin{aligned}
W^{\pm}_{\mu} &= \frac{1}{\sqrt{2}}(A^1_{\mu} \mp iA^2_{\mu}) \quad \text{with mass} \quad m_W = g\frac{v}{2}; \\
Z^0_{\mu} &= \frac{1}{\sqrt{g^2+g'^2}}(gA^3_{\mu} - g' B_{\mu}) \quad \text{with mass} \quad m_Z = \sqrt{g^2+g'^2}\frac{v}{2}.
\end{aligned} \tag{106}
$$

We can identify the massless field as our photon field and is orthogonal to Z^0_{μ};

$$
A_{\mu} = \frac{1}{\sqrt{g^2+g'^2}}(g' A^3_{\mu} + gB_{\mu}) \quad \text{with mass} \quad m_A = 0. \tag{107}
$$

In order to be able to write a general theory and have general fermion couplings easily identifiable, we replace specific gauge contributions by general gauge ones, i.e $\tau^a \to T^a$ and $+1/2 \to Y$. With this substitution, we then choose to write the covariant derivative in terms of general mass eigenstates:

$$
\begin{aligned}
D_{\mu} = \partial_{\mu} - i\frac{g}{\sqrt{2}}(W^+_{\mu}T^+ + W^-_{\mu}T^-) - i\frac{1}{\sqrt{g^2+g'^2}}Z_{\mu}(g^2T^3 - g'^2 Y) \\
-i\frac{gg'}{\sqrt{g^2+g'^2}}A_{\mu}(T^3+Y).
\end{aligned} \tag{108}
$$

Here we can easily identify the coefficient of A_{μ} as electric charge ,e, and as we have seen before, the electric charge quantum number given by $Q = T^3 + Y$. We can further symplify this expression by defining the weak mixing angle; essentially this is the angle that appears in a change of basis from (A^3, B) to (Z^0, A) and looks like a simple rotation

$$
\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix}, \tag{109}
$$

these matrix values are given by

$$
\cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}} \tag{110}
$$

By simple manipulation of the covariant derivate, we can see the couplings of the weak bosons in terms of the parameters of electron charge and our weak mixing angle θ_w.

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{\cos\theta_w} Z_\mu (T^3 - \sin^2\theta_w Q) - ieA_\mu Q$$
$$\text{with} \tag{111}$$
$$g = \frac{e}{\sin\theta_w}.$$

One can see now that at tree level calculations, all exchange processes between W and Z bosons can be described by three basic couplings. This covariant derivative describes the couplings of gauge bosons to fermions; however, it will be important to be precise when dealing with chirality.

5.3 The Higgs Lagrangian

Looking at Eq. (89) and (90), the introduction of this complex scalar doublet gives the theory a new physical particle: the Higgs boson, h. As we have already parametrized the scalar field in terms of the Higgs *vev*, we can identify a full description of the Higgs Lagrangian as

$$\mathcal{L}_\phi = |D_\mu\phi|^2 - \mu^2\phi^*\phi + \lambda(\phi^*\phi)^2, \tag{112}$$

where the covariant derivative is given by Eq. (64). We can then ask, what does the Higgs mass profile look like? If we expand the potential energy term about the minimum and evaluate the Lagrangian by our unitary parameterization, we have

$$\Delta\mathcal{L}_h = -\frac{1}{2}m_h^2 h^2 - \sqrt{\frac{\lambda}{2}}m_h h^3 - \frac{1}{4}\lambda h^4. \tag{113}$$

What we have done here is define a new field $h(x)$ with a mass defined by

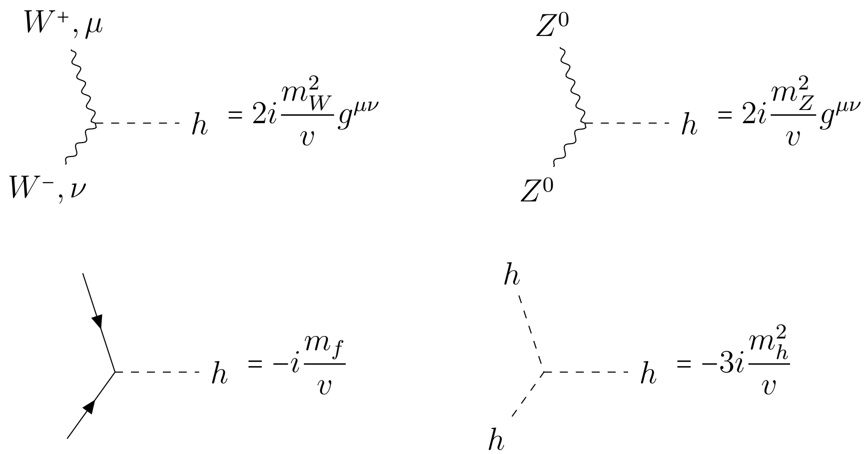
$$m_h = \sqrt{\frac{\lambda}{2}} v. \tag{114}$$

It is quite interesting, and convenient, that our mass terms for the theory depend on this value v, but are all precisely determined by their respective coupling constants (here denoted by λ). We have seen the structure of Higgs couplings to fermions given by Eq. (100) and now want to identify couplings to gauge bosons governed by the kinetic energy term in Eq. (112). This term takes the form

$$\mathcal{L}_{h,G} = \frac{1}{2}(\partial_\mu h)^2 + \left[m_W^2 W^{\mu +} W_\mu^- + \frac{1}{2} m_Z^2 Z^\mu Z_\mu\right]\left(1 + \frac{h}{v}\right)^2, \tag{115}$$

where we have already defined the masses of the gauge bosons in Eq. (106); we can also identify that there is a relationship of masses, namely $M_Z \cos\theta_W = M_W$. With these definitions and couplings, we can define Feynman rules for Higgs couplings which are given in Fig. (2). Given that couplings to the Higgs boson are proportional to the mass gauge bosons in electroweak theory, the Higgs boson is hard to detect.

Figure 2: Feynman Rules For Higgs Couplings



5.4 Higgs Phenomenology

Many properties of the Higgs boson can be probed by analysis of Higgs to two photon decay. Even though it has a very low branching ration when compared with other decays in the theory, the two photon decay process has a very clean final state along with a clear mass signature for the Higgs boson.

The Higgs boson has an incredibly short lifetime at about 10^{-22} seconds; this makes it extremely important to understand the behavior and energies of other particles in Higgs interactions. This is in essence what is done at the Large Hadron Collider (LHC); cross sections are analyzed while certain particles produced will give 'signals' to look for the Higgs. Given enough energy, the Higgs boson is created through a couple main fusion channels, these are: gluon-gluon fusion, top-top fusion, Higgs-Strahlung, W-W fusion, and Z-Z fusion. The main production mode at the LHC is gluon-gluon fusion. During these processes, the Higgs inevitably decays into various final states. Any decay that involves heavy fermions becomes difficult in measurement given the wide signal mass of the fermions compared to the very narrow width of the Higgs boson. As we now know, Higgs couplings are proportional to the mass of the respective field; thus, heavier fermions will acquire higher branching ratios.

With a predicted mass for the Higgs boson at 125 GeV, the branching ratios for each decay channel can be analyzed; these are shown in Table 2. Notice here that our decay of study has an incredible low branching ratio. The Higgs boson does not couple directly to photons, this is a loop level decay with intermediate fermions and gauge bosons. For massive vector bosons W and Z, the couplings are proportional to the square of the boson mass; whereas in the case of fermions, it is a linear relationship. This implies that the dominant mechanisms for Higgs boson production and decay involve couplings to these vector bosons.

The beauty of the specific mass and couplings of the Higgs boson is that the theory remains calculable and consistent with perturbative analysis. Values of m_H slightly greater than they are would cause the Higgs self-coupling scale to be non-perturbative at the level of quartic coupling. The width of the Higgs is proportional to the distribution of masses observed. Due to its very weak coupling, the width of the Higgs is very small compared to its center of mass. For example, with a Higgs mass of 125 GeV, it has a decay width of approximately 4 MeV.

Table 2: Branching Ratios of Higgs Boson Decays

Branching Ratios of H								
Channel	$b\bar{b}$	$\tau^-\tau^+$	W^+W^-	ZZ^*	gg	$c\bar{c}$	$\gamma\gamma$	$Z\gamma$
Br.(%)	57.7	6.32	21.5	2.64	8.57	2.91	0.228	0.154

The branching ratios given by Table 2 are calculated by a simple equation,

$$BR(h \to \sum_i f_i) = \frac{\Gamma(h \to \sum_i f_i)}{\Gamma_{TOT}}. \tag{116}$$

Where the notation just sums over any final state decay over total decay of the Higgs. To see how these decays will be calculated, it is instructive to show a very simple Higgs tree level decay; this will also illustrate our Higgs couplings.

5.5 Higgs Decay to Two Fermions

We can now analyze a general Higgs decay at tree level using the machinery we have built up. Using the appropriate Feynman diagram and rules (as according to Fig. 2), it is simple to calculate the transition amplitude, and therefore the decay width. We will look at the general process of the Higgs boson decaying into fermion/anti-fermion pairs $\mathcal{M}(h \to f\bar{f})$ (this contains the recipe for all quarks and charged leptons). When accounting for our vertex contribution and the two outgoing fermions, we have a simple amplitude of

$$\mathcal{M}_{H\to f\bar{f}} = \frac{m_f}{v}\bar{u}_{s_1}v_{s_2}, \tag{117}$$

where s_1, s_2 are the spins of the outgoing fermions. Since the decay width is our end goal, we must identify the modulus squared of the amplitude given by

$$\sum_{s_{1,2}} |\mathcal{M}|^2 = N_c \left(\frac{m_f}{v}\right)^2 \sum_{s_{1,2}} \bar{v}_{s_2}(-p_2) u_{s_1}(p_1) \bar{u}_{s_2}(p_1) v_{s_2}(-p_2). \tag{118}$$

For now, N_c is just a placeholder for an associated field's quantum number(s). We can then identify the identities that allow us to sum over the spin operators and outgoing particles as

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \quad , \quad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m. \tag{119}$$

We then apply gamma matrix identities that allow us to greatly simplify our squared amplitude. We analyze the piece of the squared amplitude that requires this attention:

$$\begin{aligned} \Delta |\mathcal{M}|^2 &= Tr(\not{p}_1 + m)(-\not{p}_2 - m) \\ &= (\gamma_\mu p_1^\mu + m)(-\gamma_\nu p_2^\nu - m) \\ &= -p_1^\mu p_2^\nu Tr(\gamma_{\mu\nu}) - m^2 Tr(1) \\ &= -4 p_1 \cdot p_2 - 4m^2 \end{aligned} \tag{120}$$

to avoid confusion, the structure of the trace is contained in the fermions spinor indices; they would be contained in a structure like $(\not{p} + m)_{\alpha\beta}$. We define the Higgs momentum by its rest mass, $q^\mu = (M_h, 0)$, we then have the relation $q^2 = (p_1 + p_2)^2$ from conservation of momentum. This can also be written as $q^2 = 2m_f^2 - 2p_1 \cdot p_2 = M_h^2$. With the same relationships of momentum and energy, we can write the final velocity of the fermions, $|p|$, as

$$p = \frac{M_h}{2}\left(1 - \frac{4m_f^2}{M_h^2}\right)^{1/2}. \tag{121}$$

We can now write our amplitude in a more conventional form as,

$$\sum |\mathcal{M}|^2 = N_c \left(\frac{m_f}{v}\right)^2 2(M_h^2 - 4m_f^2) = 2N_c \left(\frac{m_f}{v}\right)^2 M_h^2 \beta_f^2, \tag{122}$$

where we have defined

$$\beta = \left(1 - \frac{4m_f^2}{M_h^2}\right)^{1/2} = \frac{2|p|}{M_h}. \tag{123}$$

We can now calculate our decay width given, defined below, in a general sense as

$$d\Gamma = \frac{1}{2E_i} |\mathcal{M}|^2 (2\pi)^4 \delta^4 (\sum_{i,f} P_i - P_f) \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f}, \tag{124}$$

Simply plugging in the appropriate values and accounting for phase space $d\Omega$, we arrive at a very clean result of

$$\Gamma(h \to f\bar{f}) = N_c \frac{m_f^2}{v^2} \frac{M_h}{8\pi} \beta_f^3. \tag{125}$$

Here we can see the width is proportional to the mass of the Higgs but dominated by fermions mass while scaling inversely with the *vev*. The nature of perturbation theory means there are always small corrections to be done; whether these are QCD corrections (when dealing with quarks mostly) or mixing corrections, this calculation gives us a good idea of the Higgs scale in proportion to its mass. Looking at Table 2, we can now see the exact structure of the branching ratios in relation to the mass of the fields.

We can go so far as to naively calculate the partial width of the process $\Gamma(h \to b\bar{b})$. Now it is important to note that using the equation derived above is extremely naive due to a large QCD correction for quarks and running mass (as we know the couplings of the Higgs to quarks has some different properties). However, we will use this equation using some rough numbers for the bottom quark mass just to understand the scale of the calculation.

In doing so, we arrive at $\Gamma(h \to b\bar{b}) \approx 1.5$ MeV; one can understand this scale by identifying that the ratio $m_f^2/v^2 << 1$. To put this into perspective, with QCD and quark mass corrections, this value actually reaches around 2.4 MeV; this is a large discrepancy as this accounts for around a 20 % difference when calculating branching ratios. Since we know this branching ratio to be above a 50% contribution to the Higgs width, we can comfortable estimate the Higgs width to be a few MeV.

Owing to the fact that the bottom quark is incredibly heavy when compared to other quarks in the theory (besides the top quark), it receives the most corrections. If we look at the next most favored process of two tau particle decay, this only contributes a few KeV to our total decay width. The scales that we will deal with in our detailed analysis of two photon decay will even be smaller than this, but more interesting physically and computationally.

6 Higgs Boson Decay to Two Photons

Because the Higgs does not couple directly with photons, the process is mediated by fermion and W boson loops; the main contributor in the fermion case are heavy quarks, this due to greater branching ratios as a proportional to mass, as we have seen with our branching ratio percentages. The methods of solving these loop contributions are fairly technical, but can be estimated to get an order of magnitude estimate. When this is understood, it follows to analyze these processes in great detail by using dimensional regularization and loop integration techniques.

6.1 Fermion Loop Estimation

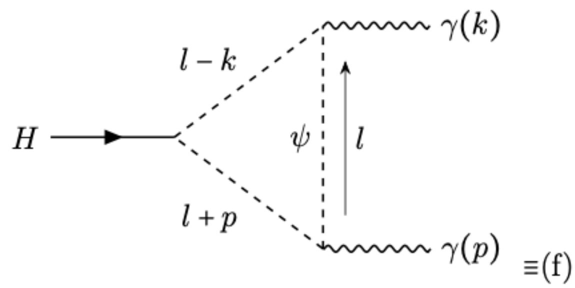
At one loop, only the heaviest fermions play a major role in the Higgs boson width. If we look at Table 2, for the purposes of two photon decay, it is completely consistent to only account for the heaviest quarks as the amplitudes will scale with the mass of the fermion. We will generalize the fermion loop calculation to illustrate the diagrams, schematics, and technique used for setting up these calculations.

Looking at Figure 3, we have a fermion loop with momentum l and our Higgs field decays at rest, denoted by its rest mass $m_h = p + k$ due to conservation of energy and momentum. We can then use the appropriate Feynman rules for Higgs-fermion vertices and fermion-photon vertices; this is given by

$$\begin{aligned} i\mathcal{M}_f = -\frac{im_f}{v} N_c Q_f^2 \int \frac{d^4 l}{(2\pi)^4} \frac{Tr\big[i(\not{l} - \not{k} + m_f)}{(l-k)^2 - m_f^2} ie\gamma^\nu \frac{i(\not{l} + m_f)}{l^2 - m_f^2} \\ \times ie\gamma^\mu \frac{i(\not{l} + \not{p} + m_f)\big]}{(l+p)^2 - m_f^2} \epsilon^\mu(p)\epsilon^\nu(k). \end{aligned} \tag{126}$$

Here we have defined the photon polarization vector by ϵ and have also defined place holders for color, N_c and charge, Q_f.

Figure 3: Fermion Loop Diagram



What we want to do now is actually take somewhat of an effective field theory approach (although we will not necessarily use the actual machinery of EFT) to estimate what our loop factor will contribute to our total amplitude and decay. A very naive choice is to look at the limit where $m_f >> m_h$; this is, in fact, a very unphysical limit and should not be taken overly serious. One can see that in this limit, as the mass of the fermion becomes large, the loop momentum will in fact decouple. This leaves us with the ability to expand the integral as $1/m_f^2$ which gives us $-ie^2/m_f^2$ to first order. Along with the solid angle factor that comes by convention when integrating, we can re-write this amplitude as

$$i\mathcal{M}_{eff} = -i(2)4\frac{m_f^2}{v}N_cQ_f^2\frac{-ie^2}{16\pi^2 m_f^2}\epsilon^\mu(p)\epsilon^\nu(k) = -\frac{8e^2}{16\pi^2 v}N_cQ_f^2\epsilon^\mu(p)\epsilon^\nu(k). \tag{127}$$

.

The factor of two comes from an identical diagram where the loop momentum is reversed; this is an invariant shift in the theory and contributes the same amplitude. the factor of $4m_f$ is factored out of the numerator terms when tracing over Dirac matrices as we have done previously. Here we have obtained an amplitude that does not depend on the momentum of the loop, l. This is obviously a crude estimation but useful to visualize the structure of these amplitudes.

6.2 W Boson Loop and $\Gamma(h \to \gamma\gamma)$ Estimation

We can also use a similar technique for W boson loops as we did for the fermion loop. Due to the necessary inclusion of Goldstone boson loops, the total amount of diagrams for the process $h \to \gamma\gamma$ is 13, the majority of which are W boson contributions. We will detail the diagrams and amplitudes specifically in the next section; however, it is instructive to see the behavior in a similar, albeit unphysical limit: $m_h << m_W$. Indeed this limit cancels most cross-diagrams and integral contributions, but leaves us with a very interesting coefficient that survives,

$$i\mathcal{M}_{eff} = 7\frac{ie^2 m_h^2}{(4\pi)^2 v}\epsilon(p)\epsilon(k). \tag{128}$$

This coefficient of '7' will become quite important when we discuss the Goldstone boson theorem as it pertains to non-decoupling of the W-loop. A simple statement of this feature is given in this limit; adding up contributions from the integrals, all terms sum to zero except the terms that are proportional to m_h^2/m_W^2. The expectation here is that in the limit we have taken, this contribution would go to zero; but of course, it does not.

Now we have very crudely calculated our Feynman amplitudes and want to pull out some actual numbers for our Higgs decay. The modulus squared of our amplitude is a sum of all contributions (fermion and W boson) and given by

$$|\mathcal{M}|^2_{eff} = \frac{\alpha^2 m_h^4}{16\pi^2 v^2}\left[8N_c Q_f^2 - 7\right]^2 \epsilon(p)\epsilon(k)\epsilon(p')\epsilon(k'). \tag{129}$$

Here we will estimate the polarization vectors by a factor of about 4 given that we assume they are proportional to $g_{\mu\nu}g^{\mu\nu}$ (this is not quite always the case, but is sufficient in our estimate). We have also defined a dimensionless constant $\alpha = e^2/4\pi$. We can then evoke Eq. (124) and integrate over phase space given by

$$d\Pi_{LIPS} = (2\pi)^4\delta^4(P_i - P_f)\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}. \tag{130}$$

Plugging in our values, we get

$$\Gamma_{eff}(h \to \gamma\gamma) = \frac{\alpha^2 m_h^3}{4\pi^2 v^2 16\pi} \left[8N_c Q_f^2 - 7\right]^2, \tag{131}$$

where the contribution from phase space is simply $1/16\pi m_h$ and dentical photons give a factor of 1/2 in the decay width. In future analysis, polarization vectors will give us terms proportional to m_h; this is due to having momentum dependence in the numerator when evaluating the integrals that contract with these vectors.

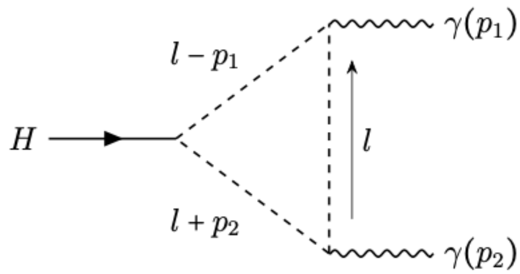
Since we know only the heaviest quarks and the W boson play a signifiant role, we can estimate the width of this channel using values for the bottom quark and W boson. Using our calculation of the width above, we find the partial decay is given by the value $\Gamma(h \to \gamma\gamma) \approx 0.0372$ MeV. As expected, this value is very small when compared to the mass of the Higgs.

If we want to see how accurate a calculation like this is, we can simply find what this value would give us for a branching ratio. Using Eq. (116), and the table above it, we can easily compare this value to that of an accepted branching ratio (taking into account that we are assuming the Higgs total width of about 4 MeV). Using our value from the estimated decay width and we arrive at a branching percentage $\mathrm{BR}(h \to \gamma\gamma) \approx 0.0093$ or .93%. As it stands now this value is actually still larger than that given our Table 2, in fact by a factor of 3. We already know this limit is highly unphysical, so the corrections when taking loop integration seriously must be of this magnitude. This can tell us two things: corrections to the fermion loop decrease its value, and/or corrections to the W loop (and Goldstones) increase its value based on their subtractive relation to each other.

6.3 $h \to \gamma\gamma$ Analysis and Results

A full analysis of Higgs decay to two photons is given in this section. For the sake of over complicating the results and interesting physics, the detailed calculations will be moved to the appendix along with techniques used to solve such problems. Here, we work in the Feynman gauge, $\xi = 1$, which simplifies our free propagators.

We begin with our fermion loop calculation, and for the sake of differentiating between our estimation previously, we will relabel our outgoing photons by p_1 and p_2 respectively (for the gauge boson loops, we will also relabel our loop momentum; as we will see the theory will not depend on loop momentums).



We start with looking at the Feynman diagram above and Eq. (126), without making any estimations as we did before. The replacement of momentums gives us a similar looking equation as previously shown:

$$\begin{aligned} i\mathcal{M}_f = &-\frac{im_f}{v} N_c Q_f^2 \int \frac{d^4 l}{(2\pi)^4} \frac{Tr\left[i(\not{l} - \not{k} + m_f)\right.}{(l-k)^2 - m_f^2} ie\gamma^\nu \frac{i(\not{l} + m_f)}{l^2 - m_f^2} \\ &\times ie\gamma^\mu \frac{\left. i(\not{l} + \not{p} + m_f)\right]}{(l+p)^2 - m_f^2} \epsilon^\mu(p_2)\epsilon^\nu(p_1). \end{aligned} \tag{132}$$

From here, although mathematically technical, the steps are simple: we simplify the numerator by contraction of indices along with Dirac matrix identities, then evaluate the denominator using Feynman parameters and a shift of variables, lastly we use dimensional regularization to take care of loop momentum dependence and divergences. These techniques can all be found in Appendix A. Applying these techniques we arrive at an amplitude of

$$i\mathcal{M}_f = -\frac{\alpha}{4\pi v}\left(m_h^2 g^{\mu\nu} - p_2^\nu p_1^\mu\right)\epsilon_\mu(p_1)\epsilon_\nu(p_2) I(\tau_f) N_c Q_f^2. \tag{133}$$

When introducing Feynman parameters, we pick up 'coordinate' dependent integrals instead of loop momentum; here that is of the form

$$I(\tau_f) = \int_{x=0}^{1} dx \int_0^{1-x} dz \frac{1 - 4xz}{1 - xz\frac{m_h^2}{m_q^2}}. \tag{134}$$

We have used some useful notation for our mass terms as $\tau_f = m_h^2/4m_f^2$. Notice also the term in the parenthesis is Eq. (133), this is a very important gauge invariant piece of our numerator. This is due to the issue of Lorentz invariance when the Ward identity is taken into account. This is a statement of polarization vectors; longitudinal polarizations of the photon are not permitted in the theory. Mathematically the statement is this:

$$p_1^\mu M_\mu = 0. \tag{135}$$

When estimating our loop integral in the previous section, this is the piece we drop out and is structurally important. Also if we have being picky, the amplitude is summed over all fermions that contribute to this decay process, as previously stated only the heaviest quarks play a major role.

We now need to account for the other diagrams that contribute to this amplitude; mostly depending on W boson contributions. Other than brute force Feynman diagram reading and calculation, there is one fun way to see that many of these cross-contributions of these diagrams actually cancel. This can be show by writing our propagator in the R_ξ gauge as

$$D_W(q) = \frac{-i}{q^2 - m_W^2}\left(g^{\mu\nu} - \frac{qq^\nu}{m_W^2}\right) + \frac{-i}{q^2 - \xi m_W^2}\frac{q^\mu q^\nu}{m_W^2}. \tag{136}$$

When reading off the diagrams after using this form of the propagator, one can take into account that the final answer cannot be gauge dependent; as we know a choice of gauge is done by convenience and does not affect the physics. With this in mind, it is clear than any contributions that are gauge dependent *must* cancel with other cross-terms in the theory.

With the same techniques described in the derivation of the fermion loop amplitude, we arrive at at a similar structure of amplitude given by

$$i\mathcal{M}_W = \frac{i\alpha}{4\pi v}(m_h^2 g^{\mu\nu} - p_2^\mu p_1^\nu)\epsilon_\mu(p_1)\epsilon_\nu(p_2) I_W(\tau_W), \tag{137}$$

where we have defined the variable

$$I_W(\tau_W) = \tau^{-1}[6I_1 - 8I_2 + \tau_W(I_1 - I_2) + I_3], \tag{138}$$

that is a combination of the following integrals, as evaluated from the Feynman amplitudes, as

$$
\begin{aligned}
I_1 &= \int_0^1 dx \log[1 - x(1-x)\tau_W] \\
I_2 &= 2\int_0^1 dx \int_0^{1-x} dy \log[1 - xy\tau_W] \\
I_3 &= \int_0^1 dx \int_0^{1-x} dy \frac{(8 - 3x + y + 4xy)\tau_W}{1 - xy\tau_W}
\end{aligned}
\tag{139}
$$

It is also convenient that both our fermion contribution along with our gauge boson take a nice compact form such that we can write the full amplitude as

$$
i\mathcal{M}(h \to \gamma\gamma) = \frac{i\alpha}{4\pi v}(m_h^2 g^{\mu\nu} - p_2^\nu p_1^\mu)\epsilon_\mu(p_1)\epsilon_\nu(p_2)\Bigg[Q_f^2 N_c I_f(\tau_f) - I_W(\tau_W)\Bigg]. \tag{140}
$$

Just as we have done previously, we will take the modulus squared of this term and use this in our decay width equation. Before doing so, it will be beneficial to evaluate the integrals and try to write them in a more compact way. To do so, we have used Mathematica to solve both integrals (as they both deal with polylogarithmic functions); after evaluation, we can write these integrals as

$$
\begin{aligned}
F &= I_W(\beta_W) + \sum_f N_c Q_f^2 I_f(\beta_f) \\
I_W(\beta) &= 2 + 3\beta + 3\beta(2-\beta)f(\beta) \\
I_f(\beta) &= -2\beta(1 + (1-\beta)f(\beta)).
\end{aligned}
\tag{141}
$$

Note the integrals defined in F are also redefined by evaluating the integrals in Eq. (139). Here we have introduced the mass term as $\beta = 4m_{f,W}^2/m_h^2$, where the subscript denotes which mass is in question, fermion or W boson. The function we have defined, $F(\beta)$ is given in terms of the mass parameters by

$$
f(\beta) = \begin{cases} \operatorname{arccsc}^2\left(\frac{1}{\beta^{-\frac{1}{2}}}\right) & \text{for} \quad \beta \geq 0 \\ -\frac{1}{4}\left[\ln\frac{1+\sqrt{1-\beta}}{1-\sqrt{1-\beta}} - i\pi\right]^2 & \text{for} \quad \beta < 1 \end{cases} \tag{142}
$$

These results are consistent with the literature by use of a simple trigonometric identity that relates $\text{arccsc}^2(x) = \arcsin^2(1/x)$. This would give us the same result in Marciano (et. al) [2], of $\arcsin^2(\beta^{-1/2})$. With Eq. (140), we can now write a very simple and concise form of our Higgs decay to two photon width in terms of Higgs mass and the *vev* as

$$\Gamma(h \to \gamma\gamma) = \frac{\alpha^2 m_h^3}{256\pi^3 v^2}|F|^2 \tag{143}$$

We can take this further to match the literature by introducing the Fermi constant

$$\frac{G_F}{\sqrt{2}} = \frac{1}{2v^2}, \tag{144}$$

this allows use to write our result in an identical fashion as in Marciano (et al), namely

$$\Gamma(h \to \gamma\gamma) = \left(\frac{\alpha}{4\pi}\right)^2 \frac{G_F m_h^3}{8\sqrt{2}\pi}\left|F\right|^2. \tag{145}$$

We can evaluate the precision of this theory by taking values for observables in the theory and seeing what numbers this gives us for our decay width, and more telling, our branching function. Here, we use accepted generic masses for the top quark, without running mass or loop corrections, with the beta dependent F-function. What we find is a decay width of 8.26228×10^{-6} GeV. If we then analyze this in our branching ratio in MeV, and using a full Higgs width of 4 MeV, we get a branching ratio of 0.00206 or 0.206%. If we then look at Table 2, we have come much closer to the accepted value than we did in our estimation previously (remember we had a percentage of .93% when estimating our loop contributions). If we want to compare this width to the results obtained when estimating the loop values in the beginning of this section, we can account for the extra order of magnitude (proportional to 10^{-1}) by understanding what happens when evaluating these integrals. When dealing with divergent integrals, the process of dimensional regularization gives us terms proportional to 1/16. Even if we estimate this beforehand, there are further corrections made when squaring the momentum dependent index structure.

Given our thorough calculation, we more than doubled our precision in the theory just by evaluating the one loop processes at some level of detail;

namely within 10% of accepted values. As previously stated, we can reach greater precision by taking into account contributions from all fermion loops, while also taking higher order loop calculations; this will be explored later, but in a qualitative way. We want to find which corrections fit into the theory to get us even closer to accepted values. Discrepancies of the above decay width can also be attributed to choice of values for respective masses along with evaluation of Eq. (141); these integrals have maximum and minimum angular dependence that can change results. Here, we have used general values without evaluating these maximums and minimums directly.

7 Higgs Boson Decay: Advanced Topics

We can now begin to probe the fundamentals of our Higgs boson decay process. We have briefly touched on the Goldstone boson theorem and how this pertains to a non-decoupling amplitude in the theory. These two things can be described in some finer detail. It follows to begin discussing ways of approaching these problems by use of other means; namely, dispersive methods and effective field theory. Along with this we can question how one can arrive at greater perturbative precision in the analysis done in this paper.

7.1 Goldstone Boson Equivalence Theorem

It is useful now to think deeper about some limiting cases of the theory; here we will see how our gauge bosons can be utilized. Simply put, the Goldstone-Boson Equivalence Theorem states that at high energies ($\Lambda >> m_W^2$), external longitudinal components of $W^{\pm}$ and Z bosons are equivalent to up to higher orders to Higgs-scalar theory. At his level one can replace W and Z bosons by Goldstone bosons. As we have seen when developing the theory and introduction of the Higgs, these bosons couple in similar ways to our other gauge bosons; however, they have an even simpler structure. We can follow the calculation of Marciano (et. al) [2], to understand how this replacement replicates the structure we are after.

Working in the Landau gauge ($\xi = 0$) to avoid gauge boson-scalar mixing, we have two diagrams, given by $\mathcal{M}^{(b)}$ and $\mathcal{M}^{(f)}$ in the appendix, where we can write them as $i\mathcal{M}_{GB}^{(b)} + i\mathcal{M}_{GB}^{(f)}$. What we then find for the amplitude is

$$
\begin{aligned}
i\mathcal{M}_{GB}^{(f)} &= (2)\frac{igm_h^2e^2}{2m_W}\epsilon_\mu(p_1)\epsilon_\nu(p_2)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k+p_1)^2(k-p_2)^2} \\
i\mathcal{M}_{GB}^{(b)} &= \frac{-2ie^2gm_h^2g_{\mu\nu}}{2m_W}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k+p_1)^2(k-p_2)^2}.
\end{aligned}
\tag{146}
$$

Here we have re-defined our loop momentum by k to emphasize these are strictly Goldstone Bosons. There sum now gives us

$$
i\mathcal{M}_{GB} = \frac{ie^2gm_h^2}{m_W}\epsilon_\mu(p_1)\epsilon_\nu(p_2)\int\frac{d^4k}{(2\pi)^4}\frac{k^2g^{\mu\nu}-4k^\mu k^\nu}{k^2(k^2+2k\cdot p_1)(k^2-2k\cdot p_2)}. \tag{147}
$$

We then expand the denominator in the usual way (Feynman parameters) and apply a shift in variables to give us

$$
i\mathcal{M}_{GB} = ie^2g\frac{m_h^2}{m_W}\epsilon_\mu(p_1)\epsilon_\nu(p_2)\int_0^1 2dy\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{N^{\mu\nu}}{(\ell^2-\Delta)^3}, \tag{148}
$$

where the numerator is defined by

$$
\begin{aligned}
N^{\mu\nu}(\ell) &= \ell^2g^{\mu\nu}-4\ell^\mu\ell^\nu-2p_1\cdot p_2yzg_{\mu\nu}+2p_1^\mu p_2^\nu yz, \\
\Delta &= m_h^2yz.
\end{aligned}
\tag{149}
$$

It is at this point one applies dimensional regularization, where we already see the electromagnetic gauge invariance form the structure of the two terms on the right of the numerator (when squared with polarization vectors this contributes a term proportional to the Higgs mass). As this is done in detail in the appendix, and this calculation follows in the same way, we can display the result given by Marciano (et. al):

$$
i\mathcal{M}_{GB} = \frac{\alpha}{2\pi v}\epsilon_\mu(p_1)\epsilon_\nu(p_2)(g^{\mu\nu}p_1\cdot p_2-p_2^\mu p_1^\nu). \tag{150}
$$

This amplitude exactly replicates the structure we obtain in our analysis of the Higgs decay to two photons, keeping in mind in our system $p_1\cdot p_2 = m_h^2/2$. Here we have acquired exactly the correct amplitude contribution using the Goldstone boson equivalence theorem.

7.1.1 Non-Decoupling

What this calculation does, is give us a greater understanding into the non-decoupling of the W loop contribution to $h \to \gamma\gamma$, in the limit $\Lambda >> m_W^2$. It is because of the unique coupling of the Higgs to these Goldstone bosons that the coupling constant does not go to zero. As we will see in our breakdown of effective field theories, the various particles of the theory acquire masses that are proportional to dimensionless couplings to the Higgs *vev*. The idea of decoupling can be as simple as setting one of these constants to zero; however, there is more to it. We have seen what happens when we take the limit of $m_f >> m_h$, where we estimated the fermion loop by a vertex contribution proportional to $1/m_f^2$. If we consider a case where the Higgs boson is much heavier than the fermion, we can think of this as setting the Yukawa coupling to zero; thus, the amplitude in this limit goes to zero.

Similar to what we did previously in the small Higgs limit, we can reverse the argument and ask what happens if the Higgs boson is much heavier than the W boson. In this limit, $m_W/m_h \to 0$. This is equivalent to saying the gauge coupling goes to zero. However, just as above, we have terms proportional to $gm_w(p_1 \cdot p_2)/m_w^2$, this is also equivalent to m_h^2/v which is exactly the term we have in our amplitudes. This term does not vanish as the Higgs *vev* is the only fundamental energy scale of the theory (In electroweak theory). This non-decoupling of the W loop is a direct consequence of the theory and is nicely checked by the Goldstone boson equivalence theorem.

7.2 Dispersive Methods and Relations

We will now describe a non-perturbative approach to solving these scattering amplitudes following from analytic properties of S-matrix. This is done through the relationship between real and imaginary portions of the matrix with the use of Cauchy's theorem. If we view our fields as two-point correlations functions, we have analytic functions proportional to the momentum, p^2. Using properties of imaginary integrals and functions, we can take intermediate branch cuts. If we take Cauchy's theorem seriously , we have a discontinuity on the positive real axis and integrate as described below. Using the optical theorem (described below), we can relate this imaginary piece of the amplitude to a sum of contribution from intermediate state particles.

Unitarity requires our S-matrix has the properties $SS^\dagger = 1$ where $S = \mathbb{I} + iT$. This allows us to write $-i(T - T^\dagger) = T^\dagger T$. We can generalize the optical theorem for two-particle states as

$$\langle p_1 p_2 | T^\dagger T | k_1 k_2 \rangle = \sum_n \left(\prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \langle p_1 p_2 | T^\dagger | \{q_i\} \rangle \langle \{q_i\} | T | k_1 k_2 \rangle. \quad (151)$$

Here we know the T-matrix elements of the equation give us a structure proportional to $-i[\mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 k_1 k_2)]$, from the relationship of the T matrix above. We now have an object that can be written as

$$\begin{aligned} &-i[\mathcal{M}(k_1 k_2 \to p_1 p_2) - \mathcal{M}^*(p_1 p_2 \to k_1 k_2)] \\ &= \sum_n \left(\prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \mathcal{M}^*(p_1 p_2 \to q_i) \mathcal{M}(k_1 k_2 \to q_i) \\ &\times (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q_i) \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \end{aligned} \quad (152)$$

We can write the integral relation above as an integral of final states of a term $M^* M$. In the case this relates to forward scattering, we can then write

$$\mathrm{Im}\mathcal{M}(k_1, k_2 \to k_1, k_2) = 2E_{cm} p_{cm} \sigma_{total}(k_1 k_2 \to \sum_f p_f) \quad (153)$$

Here we have generalized specific values; for example, momentum and cross section. This is a general form of the optical equation and relates our imaginary amplitude element to the total cross section of final states; this theorem will be necessary in understanding Feynman Diagrams in a non-perturbative way.

What follows now is an explanation and derivation that shows how we can use these ideas to describe Feynman amplitudes. The derivation follows from Peskin and Schroeder [1], defining a complex variable, s. We now have an amplitude defined as $\mathcal{M}(s)$, and define a variable s_0 as a threshold energy of a least mass multi-particle state; here for real s, $s < s_0$ implies that $\mathcal{M}(s)$ is real.

$$\mathcal{M}(s) = \left[\mathcal{M}(s^*) \right]^*. \quad (154)$$

As we saw before, there is a branch cut along the real axis, this starts at s_0 and creates a discontinuity which is given by the relation

$$\text{Disc}\mathcal{M}(s) = 2i\text{Im}\mathcal{M}(s + i\epsilon). \tag{155}$$

It is important to understand using this relation assumes on-shell virtual particles; if not, we would lose our prescription of $i\epsilon$ and our integrals would be purely real. This idea is illuminated by looking at an example, specifically in ϕ^4 theory given by Eq. (75). We have an imaginary contribution to order λ^2 which has the amplitude contribution of

$$i\delta\mathcal{M} = \frac{\lambda^2}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{(k/2 - q)^2 - m^2 + i\epsilon}\frac{1}{(k/2 + q)^2 - m^2 + i\epsilon}. \tag{156}$$

Now, we could use Feynman Parameters as before; however, we would like to verify our expression for the optical theorem. Working in the center of mass (COM) frame, we have poles at

$$q^0 = k^0/2 \pm (E_q - i\epsilon), \qquad q^0 = -k^0/2 \pm (E_q - i\epsilon). \tag{157}$$

Relative to the real q^0 axis, two of these poles lie above and two below. Peskin and Schroeder show that if the contour integral is closed below the line, only one pole contributes to the discontinuity and is is the same as writing

$$\frac{1}{(k/2 + q)^2 - m^2 + i\epsilon} \to -2\pi i\delta((k/2 + q)^2 - m^2). \tag{158}$$

This result leads to the amplitude contribution being

$$\begin{aligned} i\delta\mathcal{M} &= -2\pi i\frac{\lambda^2}{2}\int \frac{d^3q}{(2\pi)^4}\frac{1}{2e_q}\frac{1}{(k^0 - E_q)^2 - E_q^2} \\ &= -2\pi i\frac{\lambda^2}{2}\frac{4\pi}{(2\pi)^4}\int_m^\infty dE_q E_q|q|\frac{1}{2e_q}\frac{1}{k^0(k^0 - 2E_q)}. \end{aligned} \tag{159}$$

What is also shown here is one can replace the integral over the loop q by integrals over the outgoing particles, and replaced the propagators by delta functions. To the order of λ^2, what is shown is that

$$\begin{aligned} &\text{Disc}\mathcal{M}(k) = 2i\text{Im}\mathcal{M}(k) \\ &= i/2\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}|\mathcal{M}(k)|^2(2\pi)^4(p_1 + p_2 - k). \end{aligned} \tag{160}$$

This derivation shows that we can retrieve exactly the general optical theorem argument by simply replacing propagators with discontinuities in the form of delta functions. With this machinery in mind, we can now begin to describe and follow the arguments mad in Melnikov and Vainshtein's paper [3].

To do this, we define Cauchy's theorem for some analytic function $f(p^2)$ that exists inside the contour; specifically the contour is chosen to not cross singularities (In our case this would apply to cuts along the real axis). We then define the general Cauchy integral over the contour, γ, as

$$f(p^2) = \frac{1}{2\pi i} \int_{\gamma} \frac{ds f(s)}{s - p^2}. \tag{161}$$

From here Melnikov and Vainshtein go on to use this theorem to explain results using dispersion relations. If we want to naively use dispersion relations, we must define a form factor $F_W(s)$ as defined by Eq. (140) with a slight change in notation given by

$$F_W(s) = F_W^{\infty} + F_W(s) = 2 + 3\beta + 3\beta(2 - \beta) f(\beta). \tag{162}$$

Here we have the behavior as $s \to 0$, or $\beta \to \infty$, defined by F_W^{∞}. If we naively apply Cauchy's theorem with this factor we find an integral relation

$$F_W(s) = \frac{1}{\pi} \int_{4m_W^2}^{\infty} \frac{ds_1 \mathrm{Im}[F_W(s_1)]}{s_1 - s - i\epsilon}. \tag{163}$$

This is simply a statement of Cauchy's theorem where we have also applied the optical theorem given by the first piece of Eq. (159). This is a simple replacement where we only need to identify the imaginary piece as

$$\mathrm{Im}[F_W(s)] = \frac{3\pi}{2} \theta(1 - \beta)\beta(2 - \beta) \ln \frac{1 + \sqrt{1 - \beta}}{1 - \sqrt{1 - \beta}}. \tag{164}$$

Notice here, all we have done is analyze Eq. (141) while the step function, $\theta(1 - \beta)$, is introduced to take care of the singularity when $\beta = 1$.

Now it is important to understand what is actually being sought after in this paper. The form factor above contains our beta dependence, in other words, our mass and energy dependence of the process in question. The use of dispersion relations allow us to extract the real part of the amplitude from analysis of the imaginary part in a non-perturbative way. Until now we have used the typical S-matrix prescription along with functional integrals; this process is highly perturbative and can be thought of as a sort of power series.

With this in mind, one may think all that needs doing is evaluation of this imaginary integral. In doing so, we arrive at

$$F_W(s) = 3\beta + 3\beta(2-\beta)f(\beta). \tag{165}$$

This implies that our asymptotic form factor term F_W^{∞} is in fact zero. However, we know that this is not true due to non-decoupling of the W-loop. This tells us there must be some sort of interesting behavior to add to the imaginary part of the form factor. As stated in Melnikov (et.al), the most famous example of this is that of the Dirac form factor; when taken in this limiting case, it is equal to 1.

For our case in question, we need a constant to pull out of the theory, and more specifically, the integral. This must come as a physical argument from the theory; if we remember what this missing factor of 2 actually is, the choice is actually quite simple. If we choose to take the limit $s \to 0$, then we know we can apply the Goldstone boson equivalence theorem. It follows that we define a new function

$$F_W(s) = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{ds_1 \mathrm{Im} F_W^{GB}(s_1)}{s_1 - s}, \tag{166}$$

defined by

$$\mathrm{Im} F_W^{GB}(s) = -\pi m_H^2 \frac{\beta}{s} \ln\frac{1+\sqrt{1-\beta}}{1-\sqrt{1-\beta}}. \tag{167}$$

We have seen before, that this process accounts for our longitudinal degrees of freedom associated with the W bosons. If we take our limiting case seriously, as we did in previous calculations of decay rates, then we find that as $s \to 0$ Eq. (166) becomes $-2\pi m_H^2 \delta(s)$. Plugging this in to Eq. (165) we get the result

$$-\lim_{s\to 0} F_W(s) = -2. \tag{168}$$

This gives us exactly the contribution we need to form a complete description of dispersive methods for the amplitude of Higgs decay to two photons.

7.3 Higher Order Loop Corrections

Future measurements of Higgs decays, and more specifically the two photon decay channel, will require better theoretical predictions with an increase in accuracy. Higher loop QCD corrections are made at the level of 4 loops, N^3LO (next-to-next-to-next-to-leading order, as stated in the literature). It is argued that this level of correction is, in fact, not negligible and of the same order of NNLO. Perturbative corrections come in at the level of two-loop corrections; this contains running masses of quarks along with gluon emission (jets). We now follow the computational work of Davies (et. al) [4], to the correction of Higgs boson decay to two photons at four loops. From these results, we interpret the data and try to understand the results.

These calculations are done in the heavy top quark mass expansion (i.e $2m_t > m_H$). This work covers the points of theoretical uncertainty and also percentage of full decay width (the focus is on top-quark contribution, as the other fields are truly negligible). As we have seen previously, the amplitude of this process take the form

$$\mathcal{M}_{h\to\gamma\gamma} = \epsilon_1^\mu \epsilon_2^\nu (k_1 \cdot k_2 g^{\mu\nu} - k_2^\mu k_1^\nu) A(s), \tag{169}$$

here we have used the notation of the literature where $A(s)$ can be thought of as a form factor, containing information about center of mass and internal particles. In our paper A(s) has been calculated at the level of one-loop and given by Eq. (140). The focus of Davies paper is the virtual contribution of the top quark mass to the gluon-gluon-Higgs form factor, denoted as

$$A_t = \tilde{A}_t \left(Q_t^2 A_{t,t} + Q_t \sum_f Q_f A_{t,f} + \sum_f Q_f^2 A_{f,f} \right), \tag{170}$$

where $\tilde{A}_t$ is just a collection of constants proportional to the Fermi constant. This is just our fermion (top quark) contribution to the form factor taken to higher order. This is done computationally, as there are 5062 four-loop Feynman diagrams. It is important to quote the result of NNLO real corrections, as it will be combined with the corrections made in the analysis of the paper. This is given by

$$\Gamma_{h\to\gamma\gamma gg} = \frac{m_H^3}{64\pi}\tilde{A}_t\left(\frac{\alpha_s^{(5)}}{\pi}\right)^2\left(\frac{17}{34020000}\rho^4 + \frac{37}{136080000}\rho^5 + \frac{219759}{2240421120000}\rho^6\right), \tag{171}$$

defining $\rho = m_H^2/m_t^2$. It is important to note here that calculations are done in the on-shell scheme as well as the modified minimal subtraction scheme ($\bar{M}S$); this is just a technique used during analysis of integrals to absorb infinities in the theory. This is not necessarily important to understand the results; however, just to emphasize there are different ways to do the calculations.

When moving on to N^3LO top quark contributions, we finally pick out some interesting and unexpected physics. The paper gives results at each level by top quark contribution to the decay width as

$$\Gamma_{h\to\gamma\gamma} \times 10^6 GeV^{-1} = 9.1322 + 0.1558 + 0.0029 - 0.0031 = 9.2878. \tag{172}$$

This is written in terms of leading order through to N^3LO, respectively. What jumps out immediately is that NNLO corrections and N^3LO corrections are the same magnitude but different sign. In fact, in the on-shell scheme, they are larger than that of NNLO corrections and almost cancel each other out. Overall, the effect on the decay width is extremely small (average of 0.030%); however, future calculations will need precise theoretical data as experiments become more precise.

7.3.1 Two-Loop Corrections

To understand how these higher order correction are actually obtained and how they should be thought about, we can take a more detailed look at NLO corrections to the results we have derived in this paper. First, we can look at QCD corrections via virtual gluon exchange inside a quark loop (quark triangle diagram with gluon exchange). Due to charge conjugation and color conservation, single gluons cannot radiate; thus, corrections are a simple rescale of our quark contributions. This correction depends on the ratio of Higgs and quark masses. Where in our solutions previously, we have parameterized our quark loop contributing factors as in Eq. (140), we now have the parameterization

$$\tilde{F}_Q(\tau_Q) \to F_Q(\tau_Q) \times \left[1 + C(\tau_Q)\frac{\alpha_s}{\pi}\right]. \tag{173}$$

What follows is an overview a dissection of analytical analysis given by references [5,6]; however, we have changed the notation to match the results posted in this paper while specifying that we are in fact dealing with quarks; this given by the subscript 'Q'. The extra factor here is an expansion in terms of our mass factor, $\tau = 1/\beta$ as defined after Eq. (143). the coefficient is given by the expansion

$$C(\tau_Q) = c_1(\tau) + c_2(\tau)\log\frac{\mu_Q^2}{m_Q^2}, \tag{174}$$

were the coefficients c depend only on τ. As stated before, τ is defined by the ratio of the masses, however, the running quark mass evaluated at the renormalization scale is now used: i.e $m_Q(\mu_Q^2)$ (a typical selection is given by $\mu_Q = m_H/2$).

We can then analyze what happens in the heavy quark limit, also known as the low-energy theorem. We are given that in the limit $m_H^2 << 4m_Q^2$, $C(\tau_Q) \to -1$. This produces an effective Lagrangian at NLO of

$$\mathcal{L}_{eff} = e^2\frac{\alpha}{2\pi}F^{\mu\nu}F_{\mu\nu}\frac{h}{v}\left[1 - \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right]. \tag{175}$$

In the region of intermediate Higgs mass, QCD corrections are quite minimal; ranging on a scale of about $1-2\%$ correction to leading order results. In translation we can write it as

$$\Gamma_{h\to\gamma\gamma} \times 10^6 GeV^{-1} = \Gamma^{(LO)} + \Gamma^{(NLO)} = 9.1322 + 0.1558 \tag{176}$$

These numbers are strictly for QCD corrections. It is only in unphysical mass regimes of $m_H \approx 600$ GeV that we see a massive rescaling in terms of corrections. It also becomes more apparent that corrections are small due to subtractive terms that come from QED factors; things like light-fermion contributions, which are also quite minimal. One would imagine with this method, you could write higher order contributions into the theory as

$$\tilde{F}_Q(\tau_Q) = F_Q^{(0)} + \frac{\alpha_s}{\pi}F_Q^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 F_Q^{(2)} + \mathcal{O}(\alpha_s^3). \tag{177}$$

When taking limiting cases and expansions, one should always be cautious of the results as they can be quite unphysical approximations. As a result, higher order methods are done computationally or by use of effective field theories.

7.4 Effective Field Theory: Higgs Structure

As briefly shown earlier in this paper, effective field theories (hereby, "EFT's") are built as a useful model to describe the relevant physics of a specific process at a level beyond the standard model. In this technique one strives to only retain useful, useable information from the process in question. In instances where predictions are extremely difficult, EFT's are used to focus on relevant energy scales and degrees of freedom by using dimension 6 operators at some cutoff scale. The framework is built up by three main ingredients: Degrees of freedom, symmetries, and expansion parameters. In essence, one could argue that most theories are EFT's; in fact we have already employed a technique used by EFT's in the previous section when evaluating NLO contributions. We will touch on the basics of the theory as are many subtleties that come with building these methods, experimental and otherwise.

In building an electroweak/Higgs EFT, one must use a bottom-up approach. What this means is, we can write down all the operators for the corresponding particles in the theory (that are associated to the symmetries of the theory). We then choose some cutoff energy scale, here denoted by Λ, where we can no longer accurately describe the physics. If we want to display this in terms of a Lagrangian, this would look something like

$$\mathcal{L}_{EFT} = \mathcal{L}_{SM} + \sum_i \frac{C_i}{\Lambda^2}\mathcal{O}_i. \tag{178}$$

Here C_i are the associated Wilson coefficients; one can relate them to the standard model coupling constants, but these have heavier degrees of freedom dependence. A consequence of the Higgs field is its unique couplings; thus, we define operators of fermion fields separately as

$$\mathcal{L} = \mathcal{L}_{SM}^{(4)} + \sum_X C^X Q_X^{(6)} + \sum_f C'^f Q_f^{(6)}, \tag{179}$$

where X involves all other field of the theory. The Wilson coefficients are suppressed by factors of $1/\Lambda^2$. In the literature there are 8 defined classes of operators that contribute to $(h \to \gamma\gamma)$, they are as follows: X^3, ϕ^6, $\phi^4 D^2$, $\psi^2\phi^3$, $X^2\phi^2$, $\psi^2 X\phi$, $\psi^2\phi^2 D$, ψ^4. These are respectively, gauge field strength tensor X, the Higgs doublet ϕ, covariant derivative D, and a general fermion field ψ. In the building of these Lagrangians there are of course hermitian conjugations that we have neglected here. This gives us 16 operators and two extra CP-conserving operators, for a total of 18. Due to the restructure of the Lagrangian, measured quantities of the theory are rescaled in terms of new couplings based on Wilson coefficients. Consequently, the W boson, Z boson, and Higgs mass are all redefined in terms of our new operator's coefficients. Since we will largely focus on a qualitative approach to EFT's here, it will suffice to show what this would look like;

$$\begin{gathered} M_W = \frac{1}{2}\bar{g}v \\ M_Z = \frac{1}{2}\sqrt{\bar{g}^2 + \bar{g}'^2}\,v\left(1 + \frac{\bar{g}\bar{g}' C^{\phi WB} v^2}{\bar{g}^2 + \bar{g}'^2} + \frac{1}{4}C^{\phi D} v^2\right) \\ M_h^2 = \lambda v^2 - \left(3C^{\phi} - 2^{\phi} + \frac{\lambda}{2}C^{\phi D}\right)v^4, \end{gathered} \tag{180}$$

As we have seen in our previous results, the amplitudes of the process $h \to \gamma\gamma$ depends on these parameters. In this renormalized form they are also observables of the theory. Notice also that that our Wilson coefficients have indices that denote their vertex coupling dependence. Along with this, the fine structure constant is also redefined by $\bar{e}$ by

$$\bar{e} = \frac{\bar{g}\bar{g}'}{\sqrt{\bar{g}^2 + \bar{g}'^2}}\left(1 - \frac{\bar{g}\bar{g}'}{\sqrt{\bar{g}^2 + \bar{g}'^2}} C^{\phi WB} v^2\right) \tag{181}$$

The question then becomes, how can one improve the one-loop calculation for $\Gamma(h \to \gamma\gamma)$ using these techniques? Hartmann and Trott [5], outline this problem and show corrections to our one-loop process with the use of dimension 6 operators. This improvement on the theory is given by an effective Lagrangian with the structure

$$\mathcal{L}_{eff,(6)} = \sum_i C_i \mathcal{O}^i. \tag{182}$$

In this analysis, the operators of most importance to us are given by

$$\begin{aligned} \mathcal{O}_{HB} &= g_1^2 H^\dagger H B_{\mu\nu} B^{\mu\nu}, \\ \mathcal{O}_{HW} &= g_2^2 H^\dagger H W^a_{\mu\nu} W_a^{\mu\nu}, \\ \mathcal{O}_{HWB} &= g_1 g_2 H^\dagger \sigma^a H B_{\mu\nu} W_a^{\mu\nu}. \end{aligned} \tag{183}$$

The Background Field Method is then applied to pull out interesting physics in our problem. This comes in the form o

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} i \phi^= \\ h + v + \delta v + i\phi_0, \end{pmatrix} \tag{184}$$

where we parametrize our Higgs field into quantum and classical states. Here the $v + \delta v$ is the one loop classical background field and is also defined later as $\bar{v}$; however, this will not be extremely important for our calculation. We can write down our effective Lagrangian relevant to the process $\Gamma(h \to \gamma\gamma)$ at tree level. Since we already know our higher dimensional operator structure, our effective Lagrangian is written as follows:

$$\begin{aligned} \mathcal{L}_{eff} = \frac{1}{2}(h^2 + 2h\sqrt{Z_h}(v\sqrt{Z_h} + \delta v) + \phi_0^2)(\mathcal{N}_{HB} + \mathcal{N}_{HW} - \mathcal{N}_{HWB}) e^2 A_{\mu\nu} A^{\mu\nu} \\ + (\phi_+ \phi_-)(C_{HB} + C_{HW} + C_{HWB}) e^2 A_{\mu\nu} A^{\mu\nu}. \end{aligned} \tag{185}$$

The first line of the Lagrangian are the corresponding operator counter terms; these apply the appropriate renormalization constraints while taking care of divergences. These counter terms are given simply by use of a matrix that depends on our gauge couplings,

$$\mathcal{O}_i = Z_{i,j} \mathcal{O}_j. \tag{186}$$

where $Z_{i,j}$ are given in terms of gauge couplings.

Looking at the effective Lagrangian, one can simply read off the relevant tree level contributions. We can use a general definition of any decay amplitude based on calculations we have already done. This can be written as

$$\langle \phi | F(p_1) F(p_2) \rangle^0 \propto \langle \phi F^{\mu\nu} F_{\mu\nu} \rangle^0 \Big\langle \frac{\delta M_{AB}(\phi)}{\delta \phi} \Big\rangle^0, \tag{187}$$

where the first term contains our required polarization structure as dictated gauge invariance, and the second term is a mass dependent eigenstate that is proportional to the *vev.* Since we have written an effective Lagrangian to describe a one-loop contribution, we can simply read of the amplitude as

$$\langle\phi|\mathcal{A}\mathcal{A}\rangle^0_{\mathcal{L}^{(6)}} = \langle\phi\mathcal{A}^{\mu\nu}\mathcal{A}_{\mu\nu}\rangle^0\big[g_1^2 C_{HB} + g_2^2 C_{HW} - g_1 g_2 C_{HWB}\big]. \tag{188}$$

We also have defined the first term as

$$\langle\phi\mathcal{A}^{\mu\nu}\mathcal{A}_{\mu\nu}\rangle^0_{\mathcal{L}^{(6)}} = 4i[p_1^\nu p_2^\mu - (p_1 \cdot p_2)g^{\mu\nu}], \tag{189}$$

which should be recognizable from our original results. From here is is quite easy to write the full amplitude as

$$i\mathcal{A}^{\mu\nu}(h \to \gamma\gamma) = 4i[p_1^\nu p_2^\mu - (p_1 \cdot p_2)g^{\mu\nu}]\big(g_1^2 v C^{HB} + g_1^2 v C^{HW} - g_1 g_2 v C^{HWB}\big). \tag{190}$$

The gauge couplings here are defined in the usual way,

$$g_1 = \cos\theta_W \quad , \quad g_2 = \sin\theta_W. \tag{191}$$

If we define our original standard model result from Eq. (139) as

$$\langle\phi|\mathcal{A}\mathcal{A}\rangle^1_{SM}, \tag{192}$$

then we can very nicely write down our improved contribution from an EFT perspective as a full decay by

$$\Gamma_{SM+SMEFT}(h \to \gamma\gamma) = \frac{m_h^3}{4\pi}\Big|\langle\phi|\mathcal{A}\mathcal{A}\rangle^1_{SM} + \langle\phi|\mathcal{A}\mathcal{A}\rangle^0_{\mathcal{L}^{(6)}}\Big|^2. \tag{193}$$

Where the term outside of the modulus squared comes from squaring the value $\langle\phi A_{\mu\nu}A^{\mu\nu}\rangle^0$ and including the contribution from phase space. Knowing the SM result, we can use this equation to see constraints on these Wilson coefficients.

We can now ask questions about these results that lead to greater understanding. Parameterize our Wilson coefficients as proportional to $1/\Lambda^2$,we can write something in the form as follows

$$\Gamma_{SM+SMEFT}(h \to \gamma\gamma) = \frac{m_h^3}{4\pi}|SM + \frac{1}{\Lambda^2}(\cos^2\theta_W v + \sin^2\theta_W v - \cos\theta_W \sin\theta_W v)|^2. \tag{194}$$

For convenience, we have also used the relabeling

$$C^{HB} + C^{HW} - C^{HWB} = C_{\gamma\gamma} = \frac{1}{\Lambda^2} \tag{195}$$

What we can do now is ask what kind of constraints can we place on the Wilson coefficients and Λ. The reason we use EFT analysis is to make corrections to our values taken from the standard model. What we have seen from our own calculations is that we could use corrections on our decay width up to 5%.

If one simply takes a high energy scale of $\Lambda = 1$ TeV, (which is a reasonable choice given the limits of experimental energies) one finds corrections to the standard model result of about 3.14 MeV. This is a correction of almost 78% to the accepted results. If one quickly scan the literature for EFT expectations, deviations from the standard model of about $1 - 15\%$ are much more realistic. With this in mind, we can find a fit to Λ that reflects these values more accurately. What we find in doing so is an upper bound of Λ around 3 TeV and a lower bound of about 1.5 TeV (to fit these percentage corrections). This range of percentages easily covers what we would need for our results; as spoken about before, we also have corrections in other areas contained in the standard model theory.

8 Conclusion

In this paper, we have analyzed the structure and building blocks of the standard model by way of gauge theory while providing the necessary Feynman rules. We then have displayed the structure of interactions in the standard model, most specifically electroweak interactions. In doing so, we were then able to introduce symmetry breaking and the Higgs boson; all of this built up to be able to detail the calculation of Higgs boson decay to two photons at one-loop. Finally, we considered methods away from perturbation theory along with higher precision calculations in the form of higher loop analysis and EFT's.

In our detailed analysis, we found a decay width for the process $h \to \gamma\gamma$ to be 8.26228×10^{-6} GeV, or 0.00826228 MeV. Accepted values in the literature for this value are closer to 9×10^{-6} MeV. We can account for this by more detailed analysis of the term F in Eq. (144), along with corrections to quark mass, inclusion of all fermion loops, and higher order loop analysis. We have looked at these other topics in limited detail to see scale. What we also found is we can account for some loss in precision by the use of EFT models in addition to standard model results. With the ever growing need for theoretical precision, models such as these will continue to be included and tested.

Using such models, one must be careful as to not take the results too seriously. For example, having looked at dispersion relations in an attempt to understand a non-perturbative approach, one must take into account this is a delicate interplay between Goldstone's theorem and known results of the theory.

It is also worth noting that when using the Feynman gauge, one does not see the instructive cancellations between gauge dependent parameters of the R_ξ gauge, or the simplicity of diagrams in the Unitary gauge. Both of the latter gauges are much more common to work, as seen in the literature; however, the Feynman gauge makes for a more intuitive approach when analyzing the integrals.

Future work in this field, as touched on briefly, is centered around higher precision theories to match future higher energy experiments. The Higgs di-photon channel is incredibly useful and well-studied due to its clear mass signatures along with minimal 'noise'. Because of this, results of this paper are of course, not novel; however, we have replicated the results with full analysis to one loop. Furthermore, we have probed constraints on EFT models to add predictive measurements beyond the standard model. Future work in this field will be heavily influenced by EFT's as available energies and precision continue to grow. With experimental evidence of the Higgs field relatively new, future experiments will hopefully be able to shine light on either new physics, or corrections to our models.

A Appendix

A.1 Evaluation of Loop Calculations and Amplitudes

A.1.1 Numerator and Gamma Matrices: Fermion Loop

Gamma matrices are a set of matrices, $\gamma^\mu = \{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$, that ensure proper Clifford algebra for anticommutation relations for spinors; they are necessary for Dirac spin-1/2 fields in the theory. In matrix representation they are as follow:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

All group algebra is defined by their commutation or anti-commutation relationships, and gamma matrices are defined by

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I_4. \tag{196}$$

In most cases, $g^{\mu\nu}$ is generalized as the metric $\eta^{\mu\nu}$ for use of gamma matrices, for us this in not important. In practice, it will suffice to define the following identities and properties of gamma matrices:

$$
\begin{aligned}
\gamma^\mu\gamma_\mu &= 4I_4\\
Tr[\gamma^\nu\gamma^\mu] &= 4g^{\mu\nu}\\
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu\\
\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu &= 4g^{\nu\lambda}I_4\\
Tr(\gamma^\mu) &= 0\\
Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) &= 4(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda}).
\end{aligned}
\tag{197}
$$

The important note here is that *any* trace over an odd value of gamma matrices is zero.

As we have previously generalized our derivations for fermions, we will use the calculations below to speak strictly about quark loops with loop momentum l and mass m_q. We start with the same amplitude in Eq. (132) and identify the numerator as

$$
N^{\nu\mu} = (i)^5 e^2[\not{l} - \not{p}_1 + m_q]\gamma^\nu[\not{l} + m_q]\gamma^\mu[\not{l} + \not{p}_2 + m_q], \tag{198}
$$

keeping in mind that gamma matrices are position (in the equation) dependent. The expression above can be further simplified using properties of gamma matrices; to visualize this we write the numerator as

$$
\begin{aligned}
Tr[N^{\nu\mu}] = ie^2 Tr\Big[& m\{\not{l}, \gamma^\nu\}\gamma^\mu\not{p}_2 - \not{p}_1\gamma^\nu\{\not{l}, \gamma^\mu\}m\\
& + m\not{l}\gamma^\nu\{\not{l}, \gamma^\mu\} - m\not{p}_1\gamma^\nu\gamma^\mu\not{p}_2 + m\gamma^\nu\not{l}\gamma^\mu\not{l}\Big]\\
& + ie^2 m_q^3 Tr[\gamma^\nu\gamma^\mu].
\end{aligned}
\tag{199}
$$

Using the above identities, we can easily rewrite the numerator as

$$
\begin{aligned}
Tr[N^{\nu\mu}] = ie^2 4m_q\big[& (4l^\mu l^\nu - l^2 g^{\nu\mu}) + 2(l^\nu p_2^\mu - p_1^\nu l^\mu)\\
& - p_1^\nu p_2^\mu + p_1^\mu p_2^\nu + (m_q^2 - p_1\cdot p_2)g^{\mu\nu}\big].
\end{aligned}
\tag{200}
$$

We will come back to this equation after evaluation of the denominator.

A.1.2 Denominator and Feynman Parameters: Fermion Loop

We now want to see how we can manipulate the denominator in a way that always us to use dimensional regularization. We can begin with defining the denominator simply by

$$D = \frac{1}{((l-p_1)^2 - m_q^2)(l^2 - m_q^2)((l+p_2)^2 - m_q^2)}. \tag{201}$$

For the following derivation, we drop the loop integration variable dl; this will be evaluated in the next section. We will then invoke the use of Feynman Parameter. In a statement, it allows us to separate the denominator by clever use of a delta function. In mathematical terms, this is defined as

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots D_N^{\nu_N}} = \frac{\Gamma\left(\sum_{i=1}^{N} \nu_i\right)}{\prod_{i=1}^{N} \Gamma(\nu_i)} \int_0^\infty \prod_{i=1}^{N} dz_i z_i^{\nu_i - 1} \frac{\delta(1 - \sum_{j=1}^{N} z_j)}{[z_1 D_1 + z_2 D_2 + \cdots + z_N D_N]^{\sum_{i=1}^{N} \nu_i}}. \tag{202}$$

Notice here how we could easily go back to our original denominator by use of the delta function, so really we have not done anything yet. With the use of this trick, we now have a denominator written as

$$I(D) = \int_{x,y,z=0}^{1} dxdydz \frac{2\delta(x+y+z-1)}{[x((l-p_1)^2 - m_q^2) + y(l^2 - m_q^2) + z((l+p_2)^2 - m_q^2)]^3}. \tag{203}$$

We then expand the terms such that we can use the delta function in a systematic way, namely

$$I(D) = \int_{x,y,z=0}^{1} dxdydz \frac{2\delta(x+y+z-1)}{[(x+y+z)l^2 + xp_1^2 - 2xlp_1 + 2zlp_2 + zp_2^2 - (x+y+z)m_q^2]^3}. \tag{204}$$

One can see that use of the delta function will simplify this expression greatly; by doing just this we have a more compact function

$$I(D) = \int_{x=0}^{1} \int_{z=0}^{z=1-x} dxdz \frac{2}{[l^2 + xp_1^2 + 2l(zp_2 - xp_1) + zp_2^2 - m_q^2]^3}, \tag{205}$$

notice also that we lose our y dependence, this is a feature of Feynman parameters that makes it such a useful tool. Seeing that we can complete the square, we structure the denominator as

$$\begin{aligned} I(D) =& \int_{x=0}^{1} \int_{z=0}^{z=1-x} \frac{2dxdz}{[l^2 + xp_1^2 + 2l(zp_2 - xp_1) + (zp_2 - xp_1)^2 - (zp_2 - xp_1)^2 + zp_2^2 - m_q^2]^3} \\ =& \int_{x=0}^{1} \int_{z=0}^{z=1-x} dxdz \frac{2}{[(l + zp_2 - xp_1)^2 + xp_1^2 - (z^2p_2^2 + x^2p_1^2 - 2xzp_2p_1) + xp_2^2 - m_q^2]]^3} \end{aligned} \tag{206}$$

A feature of this particular channel that makes it quite a 'clean' decay is the fact that the outgoing photons are massless and their final states are known easily. By using the property

$$p_2^2 = p_1^2 = 0, \qquad (p_2 + p_1)^2 = m_h^2, \tag{207}$$

it is quite easy to see that the Higgs mass is only dependent on the relationship $2p_1 \cdot p_2 = m_h^2$. This allows us to know have the denominator of our integral much more compact and no longer a function of loop momentum,

$$I(D) = \int_{x=0}^{1} \int_{z=0}^{z=1-x} dxdz \frac{2}{[(l + (zp_2 - xp_1))^2 + xzm_h^2 - m_q^2]^3}. \tag{208}$$

From here we will be able to make a shift in variables and apply dimensional regularization, realizing that this shift in variables will also change the terms in our numerator.

A.1.3 Variable Shift and Dimensional Regularization: Fermion Loop

We now want to look at Eq. (155) and write this in a compact way. We make a shift in variables of

$$l + (zp_2 - xp_1) = \ell, \qquad xzm_h^2 - m_q^2 = \Delta, \tag{209}$$

such that now we have an integral given by

$$I(D) = \int_{x=0}^{1} \int_{0}^{1-x} dxdz \frac{2}{[\ell^2 + \Delta]^3}. \tag{210}$$

We now have to adjust the numerator because of our shift in variables to ℓ; while also utilizing the photon momentum properties of $p_1^2 = \mathrm{p}_2^2 = 0$. We write the numerator as

$$\begin{aligned} N^{\mu\nu} = 4[&(\ell^\mu \ell^\nu + \ell^\mu(xp_1^\nu - zp_2^\nu) + \ell^\nu(xp_1^\mu - zp_2^\mu) + (xp_1^\nu - zp_2^\nu)(xp_1^\mu - zp_2^\mu) \\ &- (\ell^2 + 2\ell(xp_1 - zp_2) - 2xzp_1p_2)g^{\mu\nu})] \\ &+ 2(\ell^\nu p_2^\mu - zp_2^\nu p_2^\mu + xp_1^\nu p_2^\mu - p_1^\nu \ell^\mu + zp_1^\nu \ell^\mu + zp_1^\nu p_2^\mu - xp_1^\nu p_1^\mu). \end{aligned} \tag{211}$$

A nice property to invoke to simplify this expression is identifying that our integral is symmetric; consequently any term linear in ℓ will be zero in integration. Dropping these terms and factoring, we get

$$\begin{aligned} N^{\mu\nu} = ie^2 4m_q[&4\ell^\mu \ell^\nu + \ell^2 g^{\mu\nu} + (4z^2 - 2z)p_2^\nu p_2^\mu + (4x^2 - 2x)p_1^\nu p_1^\mu - 4xzp_2^\nu p_1^\mu \\ &+(2x + 2z - 4zx)p_1^\nu p_2^\mu + xz \cdot m_h^2 g^{\mu\nu} - p_1^\nu p_2^\mu + p_1^\mu p_2^\nu + (m_q^2 - p_1 \cdot p_2) \cdot g^{\mu\nu}]. \end{aligned} \tag{212}$$

We now need to re-insert our loop dependent integration variable so that we can evaluate the integral properly. With our now somewhat factored numerator and condensed denominator, this is given by

$$\int\!\!\int_{x=0}^{1} dxdz \int\frac{d^4\ell}{(2\pi)^4} ie^2 4m_q\Bigg(\frac{2\big[4\ell^\mu\ell^\nu - \ell^2 g^{\mu\nu} + (4z^2-2z)p_2^\nu p_2^\mu + (4x^2-2x)p_1^\nu p_1^\mu\big]}{[\ell^2+\Delta]^3}$$
$$+\frac{2(2x+2z-4zx)p_1^\nu p_2^\mu + 2xz\cdot m_h^2 g^{\mu\nu} - 2p_1^\nu p_2^\mu + 2p_1^\mu p_2^\nu + 2(m_q^2 - \frac{1}{2}m_h^2)\cdot g^{\mu\nu}}{[\ell^2+\Delta]^3}\Bigg) \tag{213}$$

Notice we have made the replacement $m_h^2 = 2p_2p_1$. Our main concern will be the integral containing terms in ℓ, namely

$$I(\ell) = \int \frac{d^4\ell}{(2\pi)^4}\left(\frac{8\ell^\mu\ell^\nu}{[\ell^2+\Delta]^3} - \frac{2\ell^2 g^{\mu\nu}}{[\ell^2+\Delta]^3}\right) \tag{214}$$

We also require $\mu = \nu$, without this requirement the integral also vanishes due to our linearity argument. This then gives us the relationship

$$g_{\mu\nu}\ell^\mu\ell^\nu = \frac{\ell^2}{d} g_{\mu\nu}g^{\mu\nu} \tag{215}$$

Incorporating this into our integral, we now can write

$$I(\ell) = \int \frac{d^4\ell}{(2\pi)^4}\frac{2(\frac{4}{d}-1)g^{\mu\nu}\ell^2}{[\ell^2+\Delta]^3}. \tag{216}$$

We have now written the integral in a form that allows us to apply dimensional regularization. Using a Wick rotation, $\ell^0 \to i\ell^0$, allows us to work in euclidian space due to the new dependence of $d^4\ell = d\Omega^3 d^3\ell$, where $d\Omega$ is the area of a d-dimensional sphere; by definition

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{217}$$

The integral then takes the form

$$I(\ell) = i2(\frac{4}{d}-1)g^{\mu\nu}\int\frac{d\Omega^d}{(2\pi)^d}\int_0 d\ell\frac{\ell^2\ell^{d-1}}{[\ell^2+\Delta]^3}. \tag{218}$$

This can even be manipulated further if we identify the beta function as

$$\int_0 d\ell\frac{\ell^{d+1}}{[\ell^2+\Delta]^3} = \frac{\Delta^{d/2-2}}{2}\frac{\Gamma(2-d/2)\Gamma(1+d/2)}{\Gamma(3)}, \tag{219}$$

we can adjust our integral to take the form of the above equation as follows:

$$I(\ell) = \int \frac{d^4\ell}{(2\pi)^4} \frac{2(\frac{4}{d}-1)g^{\mu\nu}\ell^2}{[\ell^2+\Delta]^3} = 2(\frac{4}{d}-1)\frac{d}{2}\frac{1}{4^{d/2}\pi^{d/2}}\Delta^{d/2-2}\frac{\Gamma(2-d/2)}{\Gamma(3)}. \tag{220}$$

It is important to make sure the behavior is defined in the area of $d = 4$; or else this technique becomes nonsensical. Introducing the variable $2\epsilon = 4 - d$, or in other words, $\Gamma(2 - d/2) = \Gamma(\epsilon)$; we find the value of the integral to be

$$2\Big(\frac{4}{d}-1\Big)\frac{d}{2}\frac{1}{4^{d/2}\pi^{d/2}}\Delta^{d/2-2}\frac{\Gamma(2-d/2)}{\Gamma(3)} = \frac{2\epsilon}{(4\pi)^2}\left(\frac{\Delta}{4\pi}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{\Gamma(3)}. \tag{221}$$

This can be evaluated using the gamma function expansion: $\Gamma(e) = \frac{1}{e} - \gamma + \mathcal{O}(e)$, with γ being the Euler-Mascheroni constant. Inserting this expansion into our equation, we have

$$\frac{2\epsilon}{(4\pi)^2}\left(\frac{\Delta}{4\pi}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{\Gamma(3)} = \frac{2\epsilon}{(4\pi)^2}\left(\frac{\Delta}{4\pi}\right)^{-\epsilon}\frac{\frac{1}{\epsilon}-\gamma}{2}, \tag{222}$$

finally giving us a result to the integral that will be evaluated in the limit $\epsilon \to 0$ as the result below:

$$I(\ell) = i2(\frac{4}{d}-1)g^{\mu\nu}\int \frac{d\Omega^d}{(2\pi)^d}\int_0 d\ell \frac{\ell^2\ell^{d-1}}{[\ell^2+\Delta]^3} = \frac{1-\epsilon\gamma}{(4\pi)^2}. \tag{223}$$

Taking our limit, we get a finite value of $\frac{i}{16\pi^2}$; keep in mind this is only the contribution from the ℓ terms in the numerator. The rest of the integral will be denoted by $I(\tau)$ (i.e no ℓ dependence).

Now we gather the rest of the terms in the numerator that are now 'coordinate' dependent and evaluate them. The integral is given by

$$I(\tau) = \int_{x=0}^{1}\int_{0}^{1-x} dxdz\ 2ie^2 4m_q[(4z^2-2z)p_2^\nu p_2^\mu + (4x^2-2x)p_1^\nu p_1^\mu + (1-4xz)p_2^\nu p_1^\mu$$
$$+(2x+2z-1-4zx)p_1^\nu p_2^\mu + (m_q^2 - \frac{1}{2}m_h^2 + xzm_h^2)g^{\mu\nu}] \times \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+\Delta]^3}. \tag{224}$$

We can easily use the same technique as before for the last term, which will give us $\frac{i}{32\pi^2\Delta}$. The integral will also be multiplied by outgoing photon polarizations, but as we have seen before, the Ward identity states $p_2^\mu \epsilon_\mu^\lambda = 0$ and

$p_1^\nu \epsilon_\nu^\lambda = 0$. Many terms in the numerator now cancel due to this requirement and we can now write out our full amplitude as

$$i\mathcal{M} = -\frac{4m_q^2}{v} e^2 N_c Q_q^2 \int\int dxdz \left(\frac{g^{\mu\nu}}{16\pi^2} + \frac{2[(1-4xz)p_2^\nu p_1^\mu + (m_q^2 - \frac{1}{2}m_h^2 + xzm_h^2)g^{\mu\nu}]}{32\pi^2\Delta} \right) \times \epsilon_\nu^\lambda(p_1) \cdot \epsilon_\mu^\lambda(p_2). \tag{225}$$

Keeping the integral in terms of x and z, we can factor further to write a clean expression for our amplitude:

$$i\mathcal{M} = -\frac{e^2}{4\pi^2 v} N_c Q_q^2 (m_h^2 g^{\mu\nu} - p_2^\nu p_1^\mu) I_f(\tau) \epsilon_\nu^\lambda(p_1) \cdot \epsilon_\mu^\lambda(p_2). \tag{226}$$

Here we have the same integral term $I(\tau)$ as in Eq. (134). This is exactly the termused to contribute to the amplitude of $h \to \gamma\gamma$ at the fermion (mainly quark) level.

A.1.4 W-Loop Diagrams and Amplitudes

The same techniques used above are used to gather amplitudes for W-loop processes below. For the sake of redundancies, the diagrams are shown with their respective amplitudes. Although the numerator structure is different, applying the same recipe as above gives us the amplitude as in Eq. (137). The diagrams for these processes are given by Figure 4 at the end of this section. The diagrams have corresponding amplitudes shown below where we have given shorthand notation for propagators written as

$$D_s(q) = \frac{1}{q^2 - m_W^2} \quad \text{and} \quad D_W(q) = \frac{g^{\mu\nu}}{q^2 - m_W^2}, \tag{227}$$

for the Goldstone boson and W boson respectively. Here we define q as our loop momentum and also defined the sum of the outgoing photon momentum $p_1 + p_2 = k$ as the Higgs rest mass. The amplitudes are then listed with respect to the diagrams given in Figure 4, in the following way:

$$\begin{aligned} i\mathcal{M}^{(a)} &= \frac{igm_W g_{\mu\nu}}{2}(-ie^2)[2g_{\alpha\beta}g_{\rho\lambda} - g_{\alpha\rho}g_{\beta\lambda} - g_{\alpha\lambda}g_{\beta\rho}]\epsilon_\mu(p_1)\epsilon_\nu(p_2) \\ &\times \int \frac{d^Dq}{(2\pi)^D} D_W(q)D_W(q-k) \end{aligned} \tag{228}$$

$$i\mathcal{M}^{(b)} = \frac{igm_h^2}{2m_W}(2ie^2 g_{\rho\lambda})\epsilon_\mu(p_1)\epsilon_\nu(p_2) \int \frac{d^Dq}{(2\pi)^D} D_s(q)D_s(q-k) \tag{229}$$

$$i\mathcal{M}^{(c)} = i\mathcal{M}^{(d)} = -4\frac{iegg_{\mu\alpha}}{2}(-ieg_{\nu\beta})\epsilon_\mu(p_1)\epsilon_\nu(p_2) \int \frac{d^Dq}{(2\pi)^D} D_s(q)D_W(q-p_2) \tag{230}$$

$$\begin{aligned} i\mathcal{M}^{(e)} &= igm_W g_{\mu\nu}(-ie^2) \int \frac{d^Dq}{(2\pi)^D} D_W(q)D_W(q-p_1)D_W(q+p_2) \\ &\times [g_{\gamma\lambda}(2q-p_1)_\mu - g_{\lambda\mu}(2p_1-q)_\gamma - g_{\mu\gamma}(q+p_1)_\lambda] \\ &\times [g_{\rho\delta}(2q+p_2)_\nu - g_{\delta\nu}(2p_2+q)_{rho} - g_{\nu\rho}(q-p_2)_\delta] \end{aligned} \tag{231}$$

$$iM^{(f)} = -\frac{igm_h^2}{2m_W}(-ie^2)\epsilon_\mu(p_1)\epsilon_\nu(p_2)\int \frac{d^Dq}{(2\pi)^D}(2q+p_2)_\nu(2q-p_1)_\mu \\ \times D_s(q)D_s(q-p_1)D_s(q+p_2) \tag{232}$$

$$iM^{(g)} = -\frac{i}{2}gm_W(-ie^2)\int \frac{d^Dq}{(2\pi)^D}(-1)(q-p_1)_\mu q^\nu \\ \times D_s(q)D_s(q-p_1)D_s(q+p_2) \tag{233}$$

$$iM^{(h)} = iM^{(i)} = (2)\frac{-ig}{2}(-iem_W g_{\mu\lambda})\epsilon_\mu(p_1)\epsilon_\nu(p_2)(-ie)\int \frac{d^Dq}{(2\pi)^D} \\ \times (q-p_1-k)_\beta[g_{\rho\delta}(2q+p_2)_\nu - g_{\delta\nu}(q+2p_2)_\rho - g_{\nu\rho}(q-p_2)_\delta] \\ \times D_W(q)D_s(q-p_1)D_W(q+p_2) \tag{234}$$

$$iM^{(j)} = igm_W g_{\alpha\beta}(-iem_W)^2\epsilon_\mu(p_1)\epsilon_\nu(p_2)\int \frac{d^Dq}{(2\pi)^D} \\ \times D_s(q)D_W(q-p_1)D_W(q+p_2) \tag{235}$$

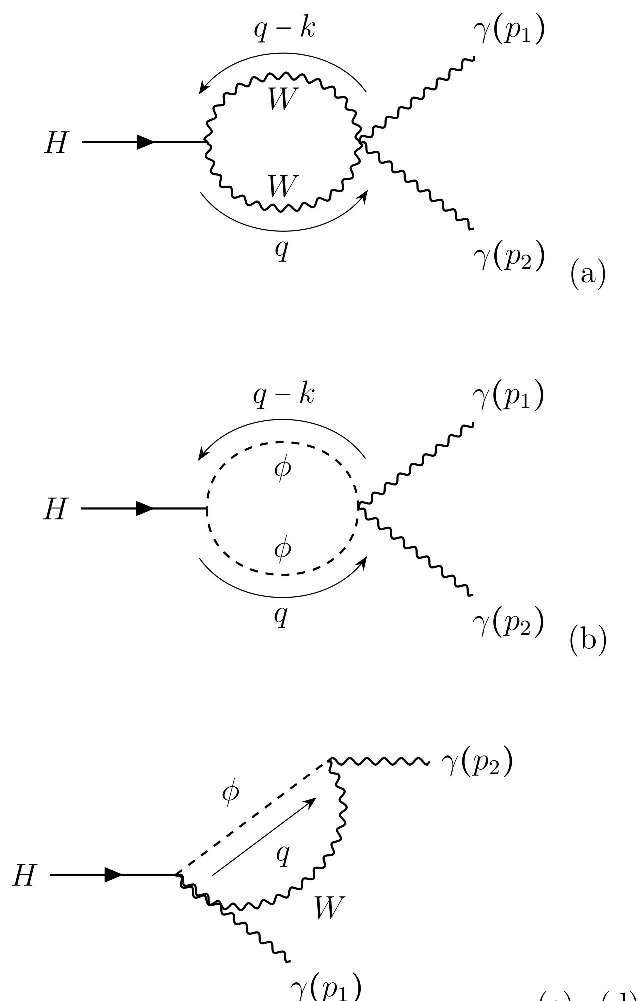
$$iM^{(k)} = iM^{(l)} = \frac{ig}{2}(-iem_W g_{\mu\gamma})(-ie)\int \frac{d^Dq}{(2\pi)^D}(p_1+2p_2+q)^\mu(2q+p_2)_\nu \\ \times D_s(q)D_W(q-p_1)D_s(q+p_2) \tag{236}$$

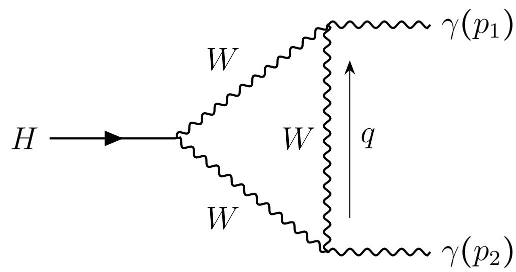
$$iM^{(m)} - \frac{igm_h^2}{2m_W}(-iem_W^2)^2\epsilon_\mu(p_1)\epsilon_\nu(p_2)\int \frac{d^Dq}{(2\pi)^D}D_W(q)D_s(q-p_1)D_s(q+p_2). \tag{237}$$

In these derivations, we have generalized the integrals immediately to D-dimensions. This is done to bring attention to the fact that they must be carefully handled as we did in section A3, and not by mathematical necessity. Along with this we have specifically written the Feynman rules in terms of the coupling constants and mass terms.

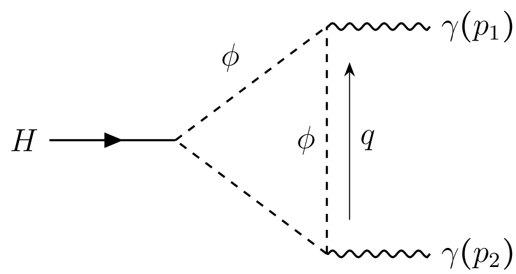
After doing so, the sum of these integral contributions indeed reproduces the result found in Eq. (138). As stated before, one can take these amplitudes in any gauge. In fact, it is quite instructive to see all the divergent and ξ dependent cancellations when working in the general R_ξ gauge. As shown in Marciano (et al) one is only left with the sum of two diagram contributions.

Figure 4: Gauge Boson Diagrams

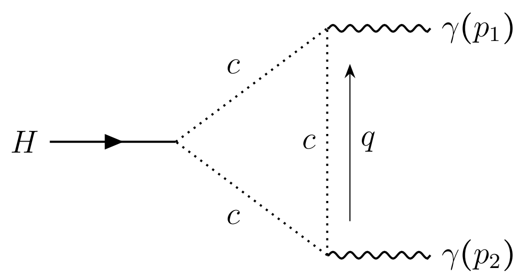



$\gamma(p_1)$
W
H
W
q
W
$\gamma(p_2)$

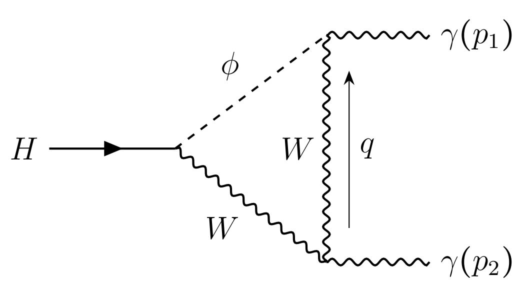
(e)


$\gamma(p_1)$
ϕ
H
ϕ
q
$\gamma(p_2)$

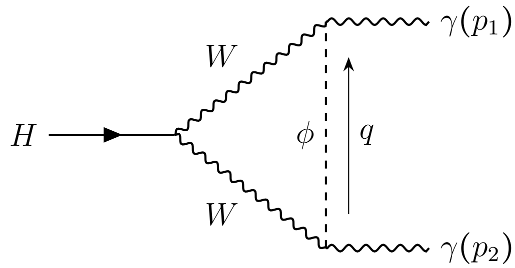
(f)


$\gamma(p_1)$
c
H
c
q
c
$\gamma(p_2)$

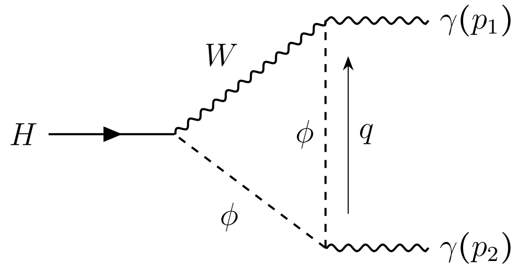
(g)


$\gamma(p_1)$
ϕ
H
W
q
W
$\gamma(p_2)$

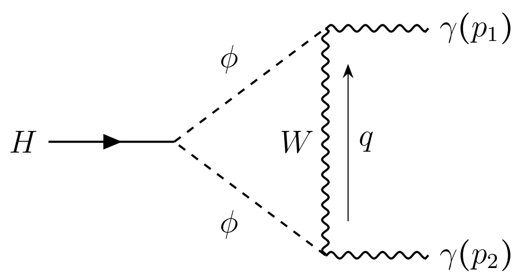
(h)=(i)


H
W
W
ϕ
q
$\gamma(p_1)$
$\gamma(p_2)$

(j)


H
W
ϕ
ϕ
q
$\gamma(p_1)$
$\gamma(p_2)$

(k)=(l)


H
ϕ
ϕ
W
q
$\gamma(p_1)$
$\gamma(p_2)$

(m)

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