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Holographic Wilson Loops

Quantum String Corrections

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Abstract

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The gauge-string duality has been one of the most exciting areas in theoretical physics as it connects strongly coupled field theories with weakly interacting strings. The present thesis concerns the study of Wilson loops in this correspondence. Wilson loops are observables arising in many physical situations like the propagation of particles in gauge fields, the problem of confinement, etc. In the gauge-string correspondence these observables have a known physical description at both sides, making them ideal probes for the duality. Remarkable progress from localization has lead to predictions at all orders in the coupling for certain Wilson loop configurations in supersymmetric field theories. Being the string theory weakly interacting, in principle we can use perturbation theory to calculate the corresponding quantities. However, current string calculations have only been successful at leading order and in rare cases, next to leading order. At next to leading order the difficulties encountered include divergences, ambiguous boundary conditions, mismatch with field theory results, etc. The research presented in this thesis aims at a better understanding of these issues. The first calculation presented here concerns the Wilson line in N=2*, a massive deformation of N=4 SYM. The string theory dual to this configuration is a straight line in the type IIB Pilch-Warner background. Using techniques for functional determinants, we computed the 1-loop string partition function obtaining perfect matching with localization. This constitutes a first test of the correspondence at the quantum level for nonconformal theories. The second calculation in this thesis corresponds to the ratio of the latitude and circular Wilson loops in AdS5xS5. An IR anomaly related to the singular nature of the conformal gauge is shown to solve previously found discrepancies with field theory results.

Keywords: Gauge-string correspondence, AdS/CFT, Wilson loops, String theory

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Dedicated to my parents and my sister

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I X. Chen-Lin, D. Medina-Rincon and K. Zarembo, *Quantum String Test of Nonconformal Holography*, JHEP 1704 (2017) 095, [arXiv:1702.07954 [hep-th]].

II A. Cagnazzo, D. Medina-Rincon and K. Zarembo, *String corrections to circular Wilson loop and anomalies*, JHEP 1802 (2018) 120, [arXiv:1712.07730 [hep-th]].

Papers not included in the thesis

III G. Arutyunov, M. Heinze and D. Medina-Rincon, *Integrability of the η-deformed Neumann-Rosochatius model*, J. Phys. A 50 (2017) no.3 035401, [arXiv:1607.05190 [hep-th]].

IV G. Arutyunov, M. Heinze and D. Medina-Rincon, *Superintegrability of Geodesic Motion on the Sausage Model*, J. Phys. A 50 (2017) no.24 244002, [arXiv:1608.06481 [hep-th]].

V D. Medina-Rincon, A. A. Tseytlin and K. Zarembo, *Precision matching of circular Wilson loops and strings in $AdS_5 \times S^5$*, arXiv:1804.08925 [hep-th].

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Contents

1. Introduction

The research presented in this thesis concerns string theory and quantum field theory in the framework of the gauge-string correspondence. This correspondence has been one of the most exciting areas in theoretical physics for the last decades and basically states the equivalence between certain quantum field theories and string theory in curved background. On one side of the correspondence we have gauge field theories, they are quantum field theories which have a "gauge group" as a symmetry, and which constitute the building blocks of the Standard Model and of our current understanding of the universe. On the other side of the correspondence is string theory, a candidate for a theory of quantum gravity and an area which has enriched various fields of mathematics over the years.

The prime example of the gauge-string correspondence is the so-called AdS/CFT duality put forward by Maldacena [1], which proposes the exact correspondence between string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM). One of the remarkable features of the proposal is that it connects two theories with different dimensionality: A string theory in five-dimensional Anti-de Sitter space and a four-dimensional field theory living in its boundary. But even more surprising is the fact that the map between the two theories is such that $\mathcal{N} = 4$ SYM at large values of the coupling constant is mapped to weakly coupled strings. This phenomenon has significant implications as some of the most challenging problems in field theory involve strong coupling, a regime where perturbation theory proves insufficient. Therefore, through this duality we can gain new insights into strongly-coupled quantum field theories by studying string theory in curved background.

At present moment the AdS/CFT duality remains a conjecture, as there is no mathematical proof despite the large amount of evidence supporting it. However, its possible implications in physics and mathematics can not be overstated as it could give us new insights into strongly-coupled phenomena like the confinement of quarks and the description of the mass spectrum of hadronic particles, a problem for which there is no theoretical explanation and is considered by the Clay Mathematical Institute to be among the seven Millennium Prize problems.

The focus of the present thesis is in further developing the tools for the string theory computation of a certain class of observables: Wilson loops. In quantum field theory Wilson loops are observables arising in many physical situations like the propagation of particles in the presence of gauge fields and are also very convenient for understanding strongly coupled phenomena, like confinement. Mathematically, they are defined through the trace of a path-ordered exponential transported along a contour C. There are several techniques for the calculation of these quantities in field theory, for instance perturbation theory or lattice simulations using computers. However, if one desires a better analytic understanding of these observables at strong coupling, a regime where perturbation theory breaks down, it is necessary to use more sophisticated techniques. Due to the strong/weak nature of the gauge-string correspondence, insights into this type of computations for a strongly coupled field theory can be achieved by computing the corresponding quantities in weakly coupled string theory.

As beautiful as this idea sounds, in practice it is very difficult as string theory in curved backgrounds has many open questions and even in perturbation theory only the first and, in very few backgrounds and configurations, second (also called 1-loop) terms have been successfully computed. In the string theory formalism, the dual to the Wilson loop is given by the path integral of a string which ends on a curve C at the boundary of AdS. At first order in the coupling parameter, contributions come from the string action evaluated at a classical solution. Meanwhile, the 1-loop term comes from the contribution of second-order fluctuations around the classical solution, a problem which in principle reduces to the evaluation of determinants of differential operators. The difficulties in this type of calculations are many, both from a technical standpoint (divergencies, possible zero-modes, choice of boundary conditions, etc.) and from a conceptual point of view as string theory has not only many fields, but also many symmetries, making the evaluation of the path integral difficult as double counting must be avoided.

Fortunately, due to remarkable progress in the field theory side coming from the use of localization and matrix models [2], there are exact predictions at all orders in the coupling constant for a few cases where the field theory and the Wilson loop have a very high degree of symmetry. These cases provide an ideal set up to deepen our understanding of how these computations should be done in string theory and serve as perfect tests for new techniques. The hope is that by learning from these computations, we can extend our knowledge of perturbative string theory calculations further and eventually make predictions for strongly coupled field theories.

The simplest Wilson loop configuration there is corresponds to a string whose geometry is a straight line in Anti-de Sitter space, also called Wilson line. This configuration has a trivial expectation value when the field theory in question is $\mathcal{N} = 4$ SYM: $\langle W \rangle = 1$. The later is due to the large amount of supersymmetry preserved by this configuration as it is 1/2 BPS, thus preserving 16 Poincaré supercharges. From the string theory perspective, this configuration was first studied for the case of $AdS_5 \times S^5$ in the pioneering work [3], where the expectation value of the Wilson line was computed up to 1-loop in string theory obtaining perfect matching with the field theory prediction.

This test of the AdS/CFT duality, as well as the related technical machinery developed for string theory in curved backgrounds, opens the door to the possibility of testing other gauge-string dualities. The quantum field theory described by $\mathcal{N} = 4$ SYM is in a way a "toy model" of the field theories describing our universe, as $\mathcal{N} = 4$ SYM has a large amount of supersymmetry and has conformal symmetry. As a "toy model", $\mathcal{N} = 4$ SYM is very useful since it is a perfect testing ground in which to test approaches to more complex theories, but at the end of the day one would like to extend current techniques to field theories closer to reality. One such theory is $\mathcal{N} = 2^*$ Yang-Mills, a cousin of $\mathcal{N} = 4$ SYM whose field content has a massive multiplet breaking conformal symmetry. This field theory has a string theory dual, the so-called Pilch-Warner background [4, 5], which is a distant cousin of $AdS_5 \times S^5$ with a considerably more complicated field content. Unlike $AdS_5 \times S^5$, the Pilch-Warner background is not integrable making calculations in this string theory a very non-trivial task.

Despite the much more complicated field content and the technical difficulties involved, in paper I we successfully calculated the 1-loop contribution to the string partition function corresponding to the Wilson line in the Pilch-Warner background and showed its divergence-free nature. By making use of methods from spectral functions, theory of differential operators and identities for isospectral operators, we managed to dramatically simplify the calculation and reproduce the field theory result of [6]. Furthermore, with these new mathematical tools in hand, we reproduced the earlier result for the Wilson line in $AdS_5 \times S^5$ [3, 7] in an elegant and relatively simple manner. The perturbative calculation in I, first of its kind for nonconformal theories, serves as a showcase for the power of spectral function methods in string theory Wilson loop calculations. Moreover, it paves the way for precision testing of the gauge-string correspondence for nonconformal theories.

The second simplest Wilson loop configuration is perhaps a circular Wilson loop in AdS, a problem whose all-loop answer is known from field theory but whose string theory calculation in $AdS_5 \times S^5$ at 1-loop has been an open question for almost a decade. In the field theory side, the result at all orders in perturbation theory was first conjectured in the foundational work [8], and then proven using localization in [2]. In the string theory side of the duality, the picture is much less clear as several attempts coming from Gel'fand Yaglom [9, 10] and heat kernel methods [7] had led to diverging results and mismatch with field theory predictions. These perplexing results suggest that perhaps the techniques used, or the way they are implemented, are not adequate for the problem at hand. Possible explanations of why past calculations have failed are also attributed to the possibility of zero-modes of the "ghost" operators appearing from the gauge fixing procedure, or ignorance on what are the right boundary conditions of these spectral problems. Furthermore, if one considers the circle to have a winding k, the situation gets even worse as previous calculations have all found different results [9, 11].

In order to leave the question of ghost zero-modes aside, whose contributions are related to the string world-sheet geometry, previous studies considered the ratio of two circular Wilson loops: the one mentioned above living entirely on AdS and at a point in S^5, and one which additionally extends in a $S^2 \subset S^5$ describing a latitude. Using Gel'fand Yaglom, previous independent perturbative 1-loop computations failed to reproduce the results from localization in field theory [12, 13], and a perturbative heat kernel approach only reproduced the first term in a series expansion of the 1-loop result for small latitude angle [14]. Recently, a similar computation using zeta function regularizarion also obtained a mismatch with the field theory prediction [15]. In paper II, using contour integration methods and the spectral function methods applied previously in the Pilch-Warner calculation of paper I, we obtained the same 1-loop result as existing Gel'fan Yaglom computations [12, 13] plus an additional contribution, successfully solving this open problem. The additional piece at the heart of the problem comes from careful consideration of the conformal transformation required for the evaluation of the functional determinants in the cylinder. This beautiful result highlights the importance and desperate need for a better understanding of the mathematical machinery required for these perturbative string theory calculations.

This thesis is organized as follows: In chapter 2 the basic string theory concepts required are introduced, in chapter 3 Wilson loops are introduced in the context of the gauge-string correspondence. Chapter 4 briefly reviews several techniques used for the computation of functional determinants. Chapters 5 and 6 concern the string theory 1-loop computation of the straight line in the Pilch-Warner background and the ratio of latitude and 1/2 BPS circular Wilson loops in $AdS_5 \times S^5$. Finally, in chapter 7 the main results are summarized and several open problems are mentioned.

2. String theory in curved backgrounds

Originally, string theory started in the 1960's as a model of hadrons but it later became apparent that this theory could describe a consistent theory of quantum gravity. General relativity, the theory which explains gravitational interactions, is (at least perturbatively) non-renormalizable which is a problematic issue as it would require the introduction of infinitely many parameters to absorb divergences. The later problem can be solved by the radical proposal of replacing point-like particles by one-dimensional objects called "strings". This proposal leads to a smoother UV behaviour and the existence of a massless spin two particle called "graviton" which interacts according to covariance laws of general relativity. Besides being a consistent theory of quantum gravity perturbatively, string theories lead to gauge groups that can include the Standard Model, consequently opening the exciting possibility of unifying gravity and the other fundamental forces under a single theoretical framework.

In addition to the mathematical consistency and possibilities of grand unification, string theory exhibits many interesting features to be studied. Among these are the existence of extra dimensions, supersymmetry and dualities. String theory has several formulations connected by an intricate web of dualities. In the present thesis we will focus exclusively on type IIB string theory as this is the one relevant for the calculations presented later. We will start by introducing this string theory in the Green-Schwarz formulation in section 2.1. Later in sections 2.2 and 2.3, we present the two supergravity backgrounds considered in our calculations: $AdS_5 \times S^5$ and the Pilch-Warner background, respectively.

2.1 Green-Schwarz type IIB superstring

A string is described by a 1+1 dimensional "worldsheet" moving in a 10 dimensional "target space" (see examples in figure 2.1). The coordinates along the worldsheet will be denoted by τ and σ, where τ is the proper time coordinate while σ is the spatial coordinate along the string. The metric tensor along the worldsheet will be denoted by h_{ij} where $\{i, j\} \in \{1, 2\}$, while in the target space the coordinates are denoted by

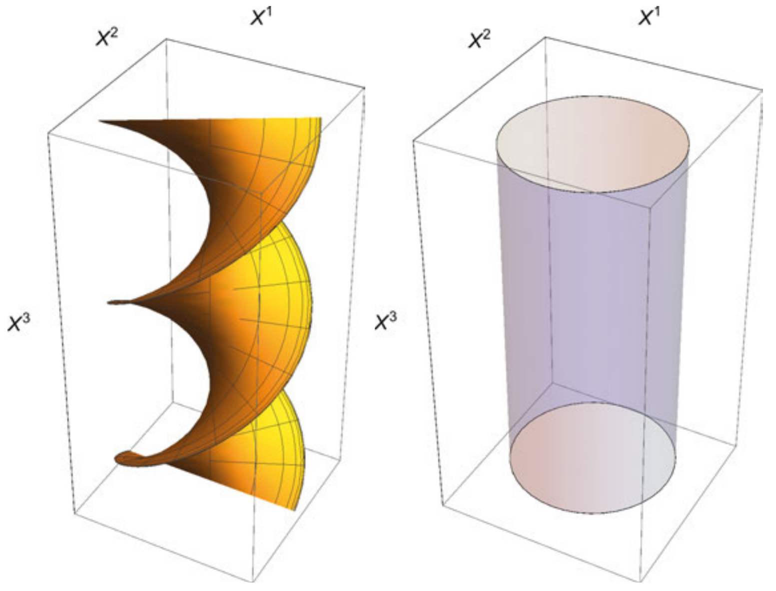

Figure 2.1. Wordsheets of an open spinning string and of a closed string.

$X^\mu = X^\mu(\tau, \sigma)$ and the metric tensor by $G_{\mu\nu}$ with $\{\mu, \nu\} \in \{1, .., 10\}$. The most general bosonic Lagrangian density is given by

$$L_{\mathrm{B}} = \frac{1}{2}\sqrt{h}h^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu} + \frac{i}{2}\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu}, \tag{2.1}$$

where the worldsheet metric is in Euclidean signature, $G_{\mu\nu}$ is the target space metric in the string frame and $B_{\mu\nu}$ is an antisymmetric B-field.

Additionally, there is a term coupling the dilaton Φ to the worldsheet metric

$$L_{\mathrm{FT}} = \frac{1}{4\pi}\sqrt{h}\, R^{(2)}\Phi, \tag{2.2}$$

where $R^{(2)}$ denotes the worldsheet Ricci scalar. The role of this term, usually referred as Fradkin-Tseytlin term, is not fully understood in the literature for the case of the Green-Schwarz string [16] but will play a key role in the computation of paper I. The bosonic action resulting from both of these contributions is

$$S_{\mathrm{B}} = \int d^2\sigma \left(\frac{\sqrt{\lambda}}{2\pi}\, L_{\mathrm{B}} + L_{\mathrm{FT}} \right). \tag{2.3}$$

The fermionic action for the Green-Schwarz string in a generic background is known perturbatively up to fourth order in fermions [17], however, for our purposes it is sufficient to consider the expression up to second order. For the type IIB superstring this fermionic piece is [16]

$$L_{\rm F}^{(2)} = \bar{\Psi}^I \left(\sqrt{h}h^{ij}\delta^{IJ} + i\epsilon^{ij}\tau_3^{\ IJ}\right) \not{E}_i \left(\delta^{JK} D_j + \frac{\tau_3^{\ JK}}{8}\partial_j X^\nu H_{\nu\rho\lambda}\Gamma^{\rho\lambda} + \frac{e^\Phi}{8}\mathcal{F}^{JK}\not{E}_j\right)\Psi^K. \quad (2.4)$$

In the expression above H is the NS-NS three-form and the fermionic field Ψ^I with $I \in \{1,2\}$ is a 32-component Majorana-Weyl spinor subject to the constraint $\Gamma^{11}\Psi^I = \Psi^I$. We use the notations $\not{E}_i = \partial_i X^\mu E_\mu{}^{\hat{\nu}}\Gamma_{\hat{\nu}}$ and $\Gamma^{\hat{\mu}_1\hat{\mu}_2\ldots\hat{\mu}_n} = \Gamma^{[\hat{\mu}_1}\Gamma^{\hat{\mu}_2}\ldots\Gamma^{\hat{\mu}_n]}$ where $E_\mu{}^{\hat{\nu}}$ is the veilbein and $\Gamma^{\hat{\mu}}$ are Dirac matrices, while D_j and $\mathcal{F}^{JK}$ are defined by [16]

$$D_j = \partial_j + \frac{1}{4}\partial_j X^\mu \omega_\mu{}^{\hat{\alpha}\hat{\beta}}\Gamma_{\hat{\alpha}\hat{\beta}} \ ,$$

$$\mathcal{F}^{JK} = \sum_{n=0}^{2}\frac{1}{(2n+1)!}\tilde{F}_{(2n+1)}^{\hat{\mu}_1\hat{\mu}_2\ldots\hat{\mu}_{2n+1}}\Gamma_{\hat{\mu}_1\hat{\mu}_2\ldots\hat{\mu}_{2n+1}}\ \sigma_{(2n+1)}^{JK} \ ,$$

where $\tilde{F}_{(i)}$ are the R-R field strengths, $\omega_\mu{}^{\hat{\alpha}\hat{\beta}}$ denotes the spin-connection and $\sigma_{(n)}$ are 2×2 matrices defined in terms of the Pauli matrices τ_i by

$$\sigma_{(1)} = -i\tau_2 \ , \qquad \sigma_{(3)} = \tau_1 \ , \qquad \sigma_{(5)} = -\frac{i}{2}\tau_2 \ .$$

In order to notationally distinguish target space indices and those of its corresponding tangent space, we have added a hat to the later. Naturally, the hatted and unhatted indices are connected by means of the vielbein $E_\mu^{\hat{\nu}}$.

The NS-NS three-form is defined by $H = dB$, while the R-R fluxes are given in terms of the superpotentials $C_{(i)}$ by

$$\begin{aligned}\tilde{F}_{(1)} &= dC_{(0)},\\ \tilde{F}_{(3)} &= dC_{(2)} + C_{(0)}\,dB,\\ \tilde{F}_{(5)} &= dC_{(4)} + C_{(2)}\wedge dB,\end{aligned}$$

where the five-form is self-dual $*\tilde{F}_{(5)} = \tilde{F}_{(5)}$ and the fluxes satisfy the Bianchi identities

$$d\tilde{F}_{(i)} = \tilde{F}_{(i-2)}\wedge dB \qquad \forall i\in\{3,5\}.$$

In order to consistently define a supergravity background in which strings propagate, one must specify its field content which for the type IIB case amounts to

$$G_{\mu\nu}, \qquad B_{\mu\nu}, \qquad \Phi, \qquad \tilde{F}_{(1)}, \qquad \tilde{F}_{(2)}, \qquad \tilde{F}_{(5)},$$

which must satisfy the supergravity equations.

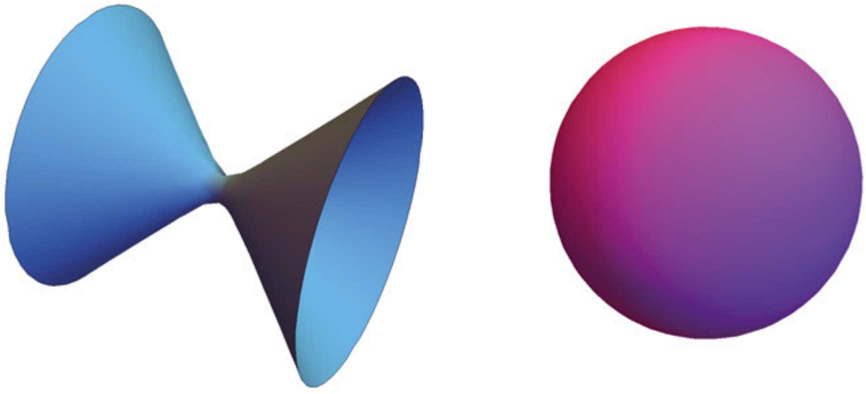

Figure 2.2. Images of anti-de Sitter space and the sphere.

Naturally, the string action resulting from equations (2.1), (2.2) and (2.4), has many symmetries including invariance under Poincaré transformations, reparametrizations of the worldsheet, Weyl transformations and κ-symmetry. These symmetries can be used to choose a convenient gauge where for instance the worldsheet metric is diagonal and the fermionic degrees of freedom are reduced in half, as will be discussed later.

2.2 Strings on $AdS_5 \times S^5$

As discussed earlier, superstrings propagate in 10 dimensional supergravity backgrounds. For type IIB supergravity the most studied example is perhaps $AdS_5 \times S^5$ due to the prominent role it plays in the AdS/CFT duality. Here, we will briefly review its geometry, field content and its symmetries.

The metric of AdS_5 in Poincaré coordinates is written as

$$ds^2_{AdS} = \frac{1}{z^2}\left(dx^2_\mu + dz^2\right), \tag{2.5}$$

where $\mu = \{1, .., 4\}$ and $z \geqslant 0$.

Meanwhile, in Hopf coordinates S^5 is described by

$$ds^2_S = d\psi^2 + \sin^2\psi\, d\varphi^2 + \cos^2\psi\left(d\phi^2_3 + \cos^2\phi_3\, d\phi^2_1 + \sin^2\phi_3\, d\phi^2_2\right), \tag{2.6}$$

where $\{\psi, \phi_3\} \in \left[0, \frac{\pi}{2}\right]$ and $\{\varphi, \phi_1, \phi_2\} \in [0, 2\pi)$.

In order to visualize these spacetimes more easily, it is sometimes convenient to use "global coordinates" given by [18]

$$X_1 + iX_2 = \sin\psi\cos\phi_3 e^{i\phi_1}, X_3 + iX_4 = \sin\psi\sin\phi_3 e^{i\phi_2}, \quad X_5 + iX_6 = \cos\psi e^{i\varphi},$$
$$Y_n = \frac{x^n}{z} \quad \forall n \in \{0,...,3\}, \qquad Y_4 = \frac{-1+z^2+x_\mu^2}{2z}, \qquad Y_5 = \frac{1+z^2+x_\mu^2}{2z}.$$

In these coordinates the metric tensors of (2.5) and (2.6) can be written as

$$\begin{aligned} ds^2_{AdS} &= \eta^{MN} dY_M dY_N \quad \text{with} \quad \eta^{MN} = (-1,+1,...,+1,-1), \\ ds^2_S &= \delta^{MN} dX_M dX_N \quad \text{with} \quad \delta^{MN} = (+1,+1,...,+1,+1), \end{aligned}$$

with the coordinates satisfying

$$\delta^{MN} X_M X_N = 1, \qquad -\eta^{MN} Y_M Y_N = 1.$$

In figure 2.2 the AdS_2 and S^2 spacetimes are presented in these coordinates.

The field content of $AdS_5 \times S^5$ is relatively simple since the only non-zero field besides $G_{\mu\nu}$ is the R-R five-form $\tilde{F}_{(5)}$. The later takes the value

$$\begin{aligned} \tilde{F}_{(5)} =& \frac{1}{z^5} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dz \\ & + \sin\phi_3 \cos\phi_3 \sin\psi \cos^3\psi \; d\psi \wedge d\varphi \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3. \end{aligned}$$

Mathematically, AdS_5 and S^5 can be seen as the cosets

$$AdS_5 = \frac{SO(4,2)}{SO(4,1)}, \qquad S^5 = \frac{SO(6)}{SO(5)},$$

where in order to include fermions one replaces the orthogonal groups with spin groups. In general, type IIB superstring theory in the $AdS_5 \times S^5$ background is a sigma-model with a target space given by the coset

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)},$$

where $SO(4,1) \times SO(5)$ is the group of local Lorentz transformations and $PSU(2,2|4)$ contains the bosonic subgroup $SO(4,2) \times SO(6)$. Naturally $\mathcal{N} = 4$ SYM, the gauge theory dual of $AdS_5 \times S^5$, has the same symmetry group.

2.3 The Pilch-Warner background

We will now present the main features and field content of the Pilch-Warner background. This supergravity background has for field theory dual $\mathcal{N} = 2^*$, which is a massive deformation of $\mathcal{N} = 4$ super Yang-Mills. Unlike $AdS_5 \times S^5$, in the Pilch-Warner background all fields of type IIB supergravity take a non-trivial value and the background itself is non-integrable due to it being much less symmetric.

The metric tensor for the Pilch-Warner background in the Einstein frame is[1] [4, 5]

$$ds_E^2 = \frac{(cX_1X_2)^{\frac{1}{4}}}{\sqrt{A}} \left[\frac{A}{c^2-1}\, dx^2 + \frac{1}{A\,(c^2-1)^2}\, dc^2 + \frac{1}{c}\, d\theta^2 + \frac{\cos^2\theta}{X_2}\, d\phi^2 + A\sin^2\theta\, d\Omega^2 \right], \tag{2.7}$$

where $c \in [1, \infty)$ and $d\Omega^2$ denotes the metric of a deformed three sphere

$$d\Omega^2 = \frac{\sigma_1^2}{cX_2} + \frac{\sigma_2^2 + \sigma_3^2}{X_1}\,. \tag{2.8}$$

In equation (2.8) the one-forms σ_i, with $i \in \{1, 2, 3\}$, satisfy

$$d\sigma_i = \epsilon_{ijk}\sigma_j \wedge \sigma_k,$$

and are defined in the $SU(2)$ group-manifold representation of S^3

$$\sigma_i = \frac{i}{2}\, \mathrm{tr}(g^{-1}\tau_i dg), \qquad g \in SU(2),$$

where τ_i are the Pauli matrices.
In equation (2.7) the functions A, X_1 and X_2 are given by

$$\begin{aligned} A &= c - \frac{c^2-1}{2}\ln\frac{c+1}{c-1}, \\ X_1 &= \sin^2\theta + cA\cos^2\theta, \\ X_2 &= c\sin^2\theta + A\cos^2\theta. \end{aligned}$$

The dilaton-axion for the Pilch-Warner background is given by

$$\mathrm{e}^{-\Phi} - iC_{(0)} = \frac{1+\mathcal{B}}{1-\mathcal{B}}\,, \qquad \mathcal{B} = \mathrm{e}^{2i\phi}\,\frac{\sqrt{cX_1}-\sqrt{X_2}}{\sqrt{cX_1}+\sqrt{X_2}}, \tag{2.9}$$

while the two-form potential $A_{(2)} = C_{(2)} + iB$ is defined as

$$A_{(2)} = e^{i\phi}\left(a_1\ d\theta \wedge \sigma_1 + a_2\ \sigma_2 \wedge \sigma_3 + a_3\ \sigma_1 \wedge d\phi\right), \tag{2.10}$$

with

$$\begin{aligned} a_1\,(c,\theta) &= \frac{i}{c}\left(c^2-1\right)^{1/2}\sin\theta\ , \\ a_2\,(c,\theta) &= i\frac{A}{X_1}\left(c^2-1\right)^{1/2}\sin^2\theta\ \cos\theta\ , \\ a_3\,(c,\theta) &= -\frac{1}{X_2}\left(c^2-1\right)^{1/2}\sin^2\theta\ \cos\theta. \end{aligned}$$

[1]Compared to the references [4, 5, 19] in our notation $A = \rho^6$. Additionally the angle θ was redefined by $\theta \to \pi/2 - \theta$.

Meanwhile, the four-form potential $C_{(4)}$ is defined as

$$C_{(4)} = 4\omega \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{2.11}$$

with

$$\omega\left(c, \theta\right) = \frac{AX_1}{4(c^2-1)^2}.$$

From the above equations it is relatively easy to compute the different type IIB R-R and NS-NS fluxes of the theory.

The Pilch-Warner geometry asymptotes to $AdS_5 \times S^5$ close to the boundary. This can be easily seen by taking the limits $c \to 1 + \frac{z^2}{2}$ and $dc \to zdz$ for small z in equation (2.7). Doing this to first order in z results in

$$ds_E^2 = \frac{dx^2 + dz^2}{z^2} + d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\Omega^2, \tag{2.12}$$

which is the usual metric of $AdS_5 \times S^5$ presented in section 2.2 (up to relabellings) with $d\Omega^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$ describing the usual three-sphere.

3. Wilson loops in AdS/CFT

The concept of Wilson loop was originally introduced by K. Wilson in the 1970's in an attempt to better understand the problem of confinement in quantum chromodynamics (QCD) in a non-perturbative manner [20], a problem that to this day is still unsolved and is one of the Millennium prize problems. Wilson loops are useful for the study of gauge field theories as these observables contain information on the underlying theory and can also play the role of order parameter in the study of confinement.

In the last decades Wilson loops have been commonly used in the study of the AdS/CFT duality as a clearer picture of Wilson loops appeared at both sides of the duality due to the D3-brane picture. The holographic dual of Wilson loops was proposed in the works [21, 22]. In the string theory picture, the dual to the Wilson loop is given by the partition function of a string embedded in AdS such that its worldsheet ends at the boundary on the contour described by the field theory loop. In the strong coupling limit, the string theory calculation of Wilson loops corresponds to the study of minimal areas of classical strings in $AdS_5 \times S^5$.

Having a holographic picture of these observables, it is in principle possible to test the AdS/CFT duality by computing Wilson loop expectation values at both sides of the correspondence (see for instance [23]). Due to advances from supersymmetric localization [2], expressions are known for several Wilson loop configurations in supersymmetric theories at all orders in the coupling. The aim of the present thesis is to further expand these tests by doing the corresponding string calculations beyond leading order.

It is also important to mention that interest in Wilson loops in AdS/CFT goes beyond testing the duality as Wilson loops over light-like polygons are conjectured to be dual to scattering amplitudes [24, 25, 26]. This surprising feature allows for the possibility of transferring knowledge from Wilson loops to the study of other observables. Study of hidden symmetries of Wilson loops is also an active area of research as the AdS/CFT duality connects Wilson loops in $\mathcal{N} = 4$ SYM to integrable classical strings on $AdS_5 \times S^5$. In this setup, progress has been made in studying Yangian symmetry at strong and weak coupling [27, 28], as well as in the study of the master and bonus symmetries [29, 30].

In this chapter, we will introduce several important physical concepts relevant for the calculations of papers I and II. Section 3.1 presents a very brief introduction to the AdS/CFT duality. In section 3.2 we review the concept of Wilson loop in field theories and in particular for the case of $\mathcal{N} = 4$ super Yang-Mills. Later, in section 3.3 the string theory picture of the Maldacena Wilson loop is presented. Finally, in section 3.4 we introduce the Wilson loop configurations studied in papers I and II, namely; the Wilson line, the circular Wilson loop and the latitude Wilson loop.

3.1 The AdS/CFT duality

In physics, dualities are of great value as they provide a bridge between a priori totally different physical theories. By providing a link between two equivalent descriptions, dualities increase our understanding of theories at a fundamental level. The gauge-string correspondence has several realizations, the AdS/CFT duality proposed by Maldacena is the classic example, but there are other dualities connecting for instance $AdS_4 \times CP^3$ and ABJM [31], or $\mathcal{N} = 2^*$ and string theory in the Pilch-Warner background.

In its "strongest version" the AdS/CFT duality states the exact equivalence of $\mathcal{N} = 4$ super Yang-Mills with $SU(N)$ gauge group and type IIB superstring theory on $AdS_5 \times S^5$. In the field theory side the parameters of the theory are the rank of the gauge group N and the value of its coupling constant $g_{\rm YM}$. Meanwhile, on the string theory side the parameters of the theory are the string coupling constant $g_{\rm s}$ and the ratio $L/\sqrt{\alpha'}$ with L denoting the radius of curvature of $AdS_5 \times S^5$ and α' being related to the string length squared [32]. According to the AdS/CFT dictionary, the parameters of these two theories are connected by

$$\lambda = g_{\rm YM}^2 N = \frac{L^4}{\alpha'^2}, \qquad\qquad g_{\rm YM}^2 = 4\pi g_{\rm s}, \tag{3.1}$$

where we introduced the 't Hooft coupling constant λ. For practical reasons it is difficult to test this duality for arbitrary values of these parameters, thus it is convenient for calculation purposes to take certain limits. Non-perturbative results in string theory are few, thus it is sometimes convenient to explore the weak coupling limit of string theory $g_{\rm s} \ll 1$ for fixed values of $L/\sqrt{\alpha'}$, which physically amounts to considering classical strings. From the r.h.s. of (3.1) we see that this corresponds to the Yang-Mills coupling $g_{\rm YM}$ being very small, and from the l.h.s. of (3.1) this would imply having λ/N to be small too. The later can be achieved by considering $N \to \infty$ for a fixed value of λ, which is referred to as the 't Hooft limit. From the field theory side all non-planar Feynman diagrams vanish in this limit and consequently it is also called the "planar limit". Thus, in this limit, classical strings in $AdS_5 \times S^5$ are dual to the planar limit of $\mathcal{N} = 4$ SYM: This is usually referred to as the "strong version" of the duality.

Having already taken the limits of $N \to \infty$ and $g_s \to 0$, there are only two arbitrary parameters left in the duality: λ in the $\mathcal{N} = 4$ SYM side and $L/\sqrt{\alpha'}$ on the $AdS_5 \times S^5$ side of the duality. As we mentioned in the introduction of chapter 1, quantum field theories at strong coupling concern some of the most challenging problems in theoretical physics, as perturbation theory breaks down. Taking the strong coupling limit corresponds to doing $\lambda \to \infty$ in the field theory side and $\sqrt{\alpha'}/L \to 0$ on the string theory side. Being L the radius of curvature of $AdS_5 \times S^5$ and α' being given by the square of the string length, $\sqrt{\alpha'}/L \to 0$ corresponds to considering strings of point-particle nature. The later are described by type IIB supergravity in $AdS_5 \times S^5$. The duality between supergravity in $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ SYM in this planar and $\lambda \to \infty$ limit is usually referred to as the "weak version" of the duality.

It is important to keep in mind the above limits as the aim of the present thesis is to study Wilson loop perturbative computations in string theory, which amounts to considering string theory in the limit $g_s \to 0$. In principle, due to localization, the Wilson loop configurations studied here are understood for arbitrary values of λ in the field theory side. However, in the string theory side only the classical contributions are fully understood, being the semiclassical contributions the object of study of the present thesis.

A natural check of the duality concerns the symmetries on both sides. As discussed in section 2.2, string theory in $AdS_5 \times S^5$ has $PSU(2,2|4)$ symmetry. In the field theory side one has the same symmetry, though it emerges in a different way. $\mathcal{N} = 4$ SYM is a conformal theory preserving $\mathcal{N} = 4$ supersymmetry: Supersymmetry in this case implies the existence of 16 Poincaré supercharges and conformal symmetry is responsible for an additional 16 supercharges, all of these symmetries form $PSU(2,2|4)$. Naturally, having the same symmetry is not a sufficient condition for the two theories to be equivalent, but it is a necessary condition.

An additional feature of the correspondence is also the fact that it realizes the holographic principle. The later is an idea in physics which states that the information contained in a d-dimensional volume is encoded in its $d-1$ boundary area. In the AdS/CFT duality the way to think about it is by considering AdS_5 as the "bulk" gravitational theory, while 4-dimensional $\mathcal{N} = 4$ SYM lives at its "boundary"; being the two theories equivalent, all the information contained in the "bulk" of AdS is encoded in its lower dimensional gauge theory dual $\mathcal{N} = 4$ SYM.

Finally, we conclude by mentioning that due to the ever increasing literature on the topic, any review on the AdS/CFT duality falls short of presenting a complete picture. However, it is worth mentioning that the duality has found applications in many areas of physics; providing new insights into relativistic hydrodynamics [33, 34], the quark-gluon plasma [35], condensed matter systems [36, 37, 38], etc. By offering the possibility of studying quantitatively strongly coupled physical phenomena, the duality is a potential window to understanding physics that would be unaccessible by other methods.

3.2 Wilson loops in field theory

In field theory, Wilson loops are operators describing the parallel transport of a very massive quark along a closed path C and physically correspond to the phase factor acquired by the quark field in such process. For the case of Yang-Mills with $SU(N)$ as gauge group, the formula describing the Wilson loop operator in the fundamental representation is [32]

$$W(C) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \left[\oint_C ds \, (i A_\mu \dot{x}^\mu) \right], \tag{3.2}$$

where $A_\mu(s)$ is the gauge field lying in the Lie algebra, $x^\mu(s)$ is a parametrization of C, P denotes path-ordering of the fields in terms of s, and the trace is taken over the fundamental representation. By definition $W(C)$ is a non-local operator and it is also gauge-invariant.

The expectation value of Wilson loop operators $\langle W(C) \rangle$, which is the physical quantity we will be interested in, plays an important role as it gives information on the field theory in question. In principle, knowledge of all Wilson loop configurations is sufficient for reconstructing the gauge potentials of the theory [39]. For instance, the quark anti-quark potential can be written in terms of the Wilson loop expectation value

$$V(R) = - \lim_{T \to \infty} \frac{1}{T} \ln \langle W(C(R,T)) \rangle ,$$

where the contour C is a rectangle of sides R and T, with $R \ll T$.

The Wilson loop operator for the case of $\mathcal{N} = 4$ super Yang-Mills was proposed by Maldacena and it is given by [21, 22]

$$W(C) = \frac{1}{N} \operatorname{tr} \mathrm{P} \exp \left[\oint_C ds \left(i A_\mu \dot{x}^\mu + |\dot{x}| \Phi_i n^i \right) \right]. \tag{3.3}$$

Compared to (3.2) the Maldacena Wilson loop introduces an extra coupling in the exponential, which depends on the scalar fields Φ_i of $\mathcal{N} = 4$ SYM and on a unit-norm six-vector $n^i(s)$ that maps every point in C to a point in S^5.

A priori such a formulation for Wilson loops in $\mathcal{N} = 4$ super Yang-Mills is non-trivial as this field theory has massless matter and transforms under the adjoint representation. To circumvent these issues the proposal in [21] starts with a $SU(N+1)$ $\mathcal{N} = 4$ SYM field theory and introduces massive quarks by means of a breaking of symmetry $SU(N+1) \to SU(N) \times U(1)$ such that the corresponding W-bosons acquire a mass and transform in the fundamental representation of $SU(N)$. This breaking of symmetry from $SU(N+1)$ is such that the scalars of this theory, which are valued in $\mathfrak{su}(N+1)$, break into the massless scalars of $\mathfrak{su}(N)$ plus fields transforming in the fundamental (and anti-fundamental) representation of $\mathfrak{su}(N)$ which have mass due to the Higgs mechanism. The phase factor in the propagator of such fields in background gauge and scalar fields is in fact the Maldacena Wilson loop [40].

3.3 Wilson loops in string theory

The string theory dual of the Maldacena Wilson loop expectation value is given by the partition function of a string whose worldsheet Σ ends on a contour C at

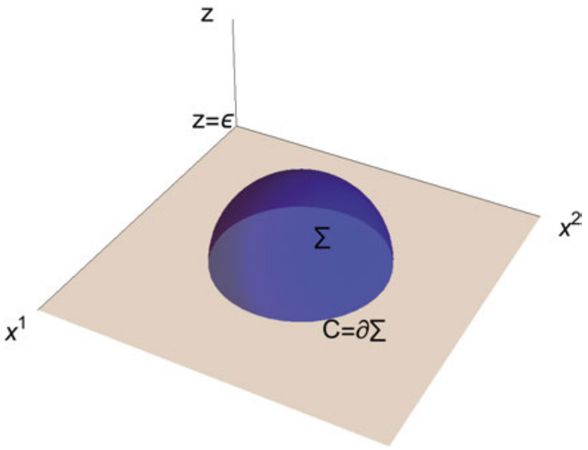

Figure 3.1. Holographic picture of a Wilson loop: The string worldsheet is in the bulk of AdS and encloses C at the boundary.

the boundary of AdS, as seen in Figure 3.1. The superstring partition function is given by [21]

$$\langle W(C)\rangle = Z = \int DX^{\mu} D\Psi Dh_{ij} \quad e^{-S_{\text{String}}(h,X,\Psi)},$$

where the integration is carried over the target space coordinates X^{μ}, the fermionic fields Ψ and the worldsheet metric h_{ij}.

At a glance it is easy to see that the expression above is not well defined as it has overcounting over physically equivalent configurations which are related by Weyl and diffeomorphism transformations. Therefore, in order to make sense of the path integral, it is necessary to do gauge-fixing. We first fix the worldsheet metric to be given by the metric induced by a classical solution ${X_{\text{cl}}}^{\mu}(\tau,\sigma)$ ending at the required geometry in the boundary of AdS

$$h_{ij} = \partial_i {X_{\text{cl}}}^{\mu}(\tau,\sigma)\ G_{\mu\nu}(X_{\text{cl}})\ \partial_j {X_{\text{cl}}}^{\nu}(\tau,\sigma)\,, \tag{3.4}$$

where the classical string solution satisfies the boundary condition

$${X_{\text{cl}}}^{\mu}(\tau,\sigma)|_{\partial\Sigma} = x^{\mu}(s)\,,$$

with the r.h.s. being previously defined in section 3.2. In order for this gauge-fixing procedure to produce the correct measure of the path integral, it is necessary to introduce the Faddeev-Popov ghosts. Additionally, it is also necessary to fix the fermionic κ-symmetry in order not to double count fermionic degrees of freedom. There are many ways to fix the later, here we do as in [3, 9] and impose $\Psi^1 = \Psi^2 = \Psi$. These considerations result in

$$\langle W(C)\rangle = Z = \int DX^{\mu} D\Psi \quad e^{-S_{\text{String}}(X,\Psi)} \quad \det{}^{1/2} P^{\dagger} P,$$

where the determinant on the right is due to the Faddeev-Popov procedure.

Following the semiclassical quantization procedure of [3] for the Green-Schwarz string, we expand the string embedding coordinates X^{μ} around the classical solution X^{μ}_{cl}

$$X^{\mu} = X^{\mu}_{\text{cl}} + \delta X^{\mu} = X^{\mu}_{\text{cl}} + E_a{}^{\mu} \xi^a, \tag{3.5}$$

where δX^{μ} denote quantum fluctuations around the classical solution and ξ^a their corresponding projection into tangent space. The later will be the default basis in which to consider bosonic fluctuations since fermions are only defined in the tangent space and, due to supersymmetry, both path integrals should be treated in a similar manner.

By expanding the bosonic Lagrangian density (2.1) up to second order in fluctuations and integrating by parts, one finds the structure $L_{\text{B}}(X) = L_{\text{B}}(X_{\text{cl}}) + \xi \mathcal{K}_B \xi$ where linear terms in ξ vanish due to the equations of motion and $\mathcal{K}_B$ denotes second order differential operators. Meanwhile, the expansion in fluctuations for the fermionic piece (2.4) results in $L_{\text{F}}(X, \Psi) = \bar{\Psi} \mathcal{K}_F \Psi$ with $\mathcal{K}_F$ denoting differential operators linear in derivatives. By replacing in the path integral it is easy to see that both fermionic and bosonic integrals are of Gaussian type, thus obtaining

$$\langle W(C) \rangle = Z = e^{-S_{\text{String}}(X_{\text{cl}})} \frac{\det^{1/2} \mathcal{K}_F}{\det^{1/2} \mathcal{K}_B} \det{}^{1/2} P^{\dagger} P. \tag{3.6}$$

The expression above for the string partition function is composed of two pieces, each one with a different functional dependence on λ. We now present the main features of each.

3.3.1 The classical action

The contribution coming from the classical action $S_{\text{String}}(X_{\text{cl}})$ is obtained by evaluating equations (2.1) and (2.2) at the classical solution. Physically, this contribution corresponds to the leading term of the Wilson loop expectation value at strong coupling ($\lambda \gg 1$) and its dependence on the 't Hooft coupling is of the form $S_{\text{String}}(X_{\text{cl}}) = \sqrt{\lambda}$ Const. In the literature, such classical contributions are well understood and in general agree with field theory predictions.

For the case of the $AdS_5 \times S^5$ background, classical contributions have a nice geometrical interpretation. Due to the absence of B-field and dilaton, the only non-vanishing contribution comes from the Polyakov action evaluated at the classical solution. The later is understood to have the following structure

$$\langle W(C) \rangle \overset{\lambda \gg 1}{=} \exp\left[-\frac{\sqrt{\lambda}}{2\pi} A_{\text{ren}}(C) \right],$$

where A_{ren} denotes the "renormalized" area of the worldsheet. In principle, direct evaluation of the Polyakov string at the classical solution results in a divergent result. The reason for this divergence resides in the existence of the

$1/z^2$ singularity of the metric tensor at the boundary. The way to regularize this divergence is by considering the integration volume to start from a small cutoff $z = \epsilon$ close to the boundary. The regularization procedure can be summarized by considering

$$A_{\text{ren}}\left(C\right) = \lim_{\epsilon \to 0} \left[A\left(C\right)\big|_{z \geqslant \epsilon} - \frac{\mathrm{L}}{\epsilon} \right],$$

where L denotes the perimeter around C. Another way to think about the regularization procedure is by implementing the Legendre transform of Z, which amounts to dropping the $1/\epsilon$ divergences in the final result [40].

It is important to keep in mind that for more complicated supergravity backgrounds with non-trivial dilaton, like the Pilch-Warner background, one has an additional contribution coming from the Fradkin-Tseytlin term. This contribution is classical in nature and will play an important role in the calculations of paper I.

3.3.2 The semiclassical partition function

Contributions to the semiclassical partition function come from the evaluation of the functional determinants in (3.6) and will be the focus of the present thesis.

In principle, provided that the functional determinants considered do not have zero modes, the semiclassical partition function would have a dependence on λ of the form λ^0. If the spectrum in one of the determinants has zero modes, each would contribute with a factor $\lambda^{-1/4}$ to the determinant. The later can be seen more clearly through the use of collective coordinates when evaluating the determinants [41].

Before proceeding, we will briefly present some features of the individual contributions to the semiclassical partition function. The fermionic contribution comes from fixing κ-symmetry and evaluating the terms in between $\bar{\Psi}$ and Ψ in (2.4) at the classical solution. In principle, being Ψ a 16-component spinor, the resulting differential operator will be a 16×16 matrix. The later can be usually reduced to much simpler 2×2 blocks by choosing a convenient representation of Dirac matrices. Naturally, the choice of gauge and Dirac matrices does not change the physics of the problem, but the expressions involved become much simpler. The resulting differential operators $\mathcal{K}_F$ will be linear in derivatives with respect to τ and σ. Since it is customary (although not necessary) to consider determinants of operators of a Laplace type $-\nabla^\mu \nabla_\mu$, the fermionic operators in $\mathcal{K}_F$ are sometimes squared and one studies the determinant of the later [42].

The bosonic contribution comes from expanding the ten coordinates in (2.1) around the classical solution. Depending on the field content involved, it is sometimes convenient to use the following expression from which one can read the differential operators after partial integration and projecting the fluctuations to tangent space [43]

$$L_{\mathrm{B}}^{(2)} = \frac{1}{2}\sqrt{h}\left[G_{\mu\nu}\left(X_{\mathrm{cl}}\right)\mathcal{D}_i\xi^\mu\mathcal{D}^i\xi^\nu + R_{\mu\nu\alpha\beta}\left(\partial_i X_{\mathrm{cl}}^\mu\right)\left(\partial^i X_{\mathrm{cl}}^\beta\right)\xi^\nu\xi^\alpha\right.$$
$$\left. -\frac{1}{2}\left(\partial_i X_{\mathrm{cl}}^\alpha\right)\left(\partial_j X_{\mathrm{cl}}^\beta\right)\varepsilon^{ij}\xi^\mu\xi^\nu\nabla_\mu H_{\nu\alpha\beta} + \frac{1}{4}\left(\partial_i X_{\mathrm{cl}}^\nu\right)\left(\partial^i X_{\mathrm{cl}}^\beta\right)H_{\mu\nu\alpha}H^\alpha{}_{\beta\lambda}\xi^\mu\xi^\lambda\right],$$

where ϵ_{ij} is the Levi-Civita tensor, ∇ is the covariant derivative, h_{ij} was defined in equation (3.4) and

$$\mathcal{D}_i\xi^\mu = \partial_i\xi^\mu + \Gamma^\mu_{\nu\alpha}\left(X_{\mathrm{cl}}\right)\left(\partial_i X_{\mathrm{cl}}^\alpha\right)\xi^\nu + \frac{1}{2}\varepsilon_i{}^j\left(\partial_j X_{\mathrm{cl}}^\nu\right)H^\mu{}_{\nu\alpha}\xi^\alpha.$$

When evaluating the functional determinants it is important to remember that each spectral problem corresponds to a differential operator, a set of boundary conditions and an inner product. The inner product is important in the evaluation of the functional determinants as the later should only have contributions from normalizable fluctuations. The norm of fluctuations (in tangent space) is given by the inner product

$$\left\langle\xi^a,\xi^b\right\rangle = \iint \sqrt{h}\ \xi^a\xi^b d\tau d\sigma.$$

The Faddeev-Popov determinant results from the Gaussian integration of the ghost action. The later is of the form [3, 44]

$$L_{\mathrm{FP}} = \frac{1}{2}\sqrt{h}h^{ij}\left(\nabla_k\varepsilon_i\nabla^k\varepsilon_j - \frac{R^{(2)}}{2}\varepsilon_i\varepsilon_j\right).$$

After projecting on the worldsheet tangent space using the worldsheet veilbein ($\varepsilon_i = e_i{}^{\hat{j}}\varepsilon_{\hat{j}}$) and integrating by parts, it can be shown that the resulting operator is the same as the bosonic differential operator along directions longitudinal to the string worldsheet.

The fact that the ghost and bosonic longitudinal modes result in the same differential operator does not imply that the resulting determinants are the same and that their contributions cancel. The reason for this is that the spectrum also depends on the boundary conditions imposed, and the later are not the same for ghosts and bosonic longitudinal modes [3]. The difference in the spectrum of ghosts and longitudinal modes is sometimes attributed for the mismatch in semiclassical string partition function calculations. In particular, presence of zero modes in one of the operators is commonly assumed to be responsible for $\ln\lambda$ terms when evaluating $\ln\langle W(C)\rangle$ [23].

In order to avoid these ambiguities we evaluate the ratio of two Wilson loops with the same geometry in paper II, where the corresponding contributions are expected to cancel between latitude and circular Wilson loops. Meanwhile, for the case of the straight line the ratio of ghost and longitudinal mode determinants is assumed to cancel. This is motivated by the fact that ghost zero modes are not normalizable in this case and that similar ghost/longitudinal mode cancelations have lead to correct results in string theory [9, 45]. Thus, from now on when mentioning bosonic operators we refer exclusively to those along the eight directions transversal to the string worldsheet.

From the considerations above, the quantity that we will study is

$$\langle W(C)\rangle = Z = e^{-S_{\rm String}(X_{\rm cl})}\frac{\det^{1/2}\mathcal{K}_F}{\det^{1/2}\ \mathcal{K}_B}. \tag{3.7}$$

In order to evaluate the determinants, we will impose Dirichlet boundary conditions at the boundary of AdS. In principle, an eigenfunction of a second order differential operator is given by the superposition of two solutions. In our procedure we choose those solutions that vanish at the boundary and are well behaved in its neighbourhood. Due to the 2×2 nature of fermionic operators, it is often the case that one component vanishes while the other does not at the boundary. For this fermionic cases we choose the superposition which is finite approaching the boundary, as usually one of them diverges. More details on the boundary conditions imposed are presented in chapters 5 and 6. The technique used to evaluate the corresponding determinants is introduced later in chapter 4.

3.4 Loop configurations

We now present the Wilson loop configurations considered in papers I and II. Their geometry and localization predictions are shown for each case, and we comment on the state of the corresponding string theory calculations.

3.4.1 The Wilson line

This Wilson loop operator corresponds to an infinite straight line with a contour parametrized by $x^\mu(s) = (s,0,0,0)$. When the field theory considered is $\mathcal{N}=4$ SYM the result is

$$\langle W(C)\rangle_{\rm line} = 1. \tag{3.8}$$

The above result is a consequence of the Wilson loop operator commuting with half of the 16 supercharges, making it BPS, and thus being protected from quantum corrections [8]. From the holographic perspective, this is perhaps the best understood configuration as the classical and semiclassical string partition functions in $AdS_5\times S^5$ have been shown to reproduce the field theory result using several methods for the evaluation of determinants [3, 7, 9]. The corresponding classical string solution is given by

$$z_{\rm cl} = \sigma, \qquad\qquad x^0_{\rm cl} = \tau, \tag{3.9}$$

while other coordinates vanish such that the classical string describes an infinitely long line in AdS with $\sigma\in[0,\infty)$ and $\tau\in[0,2\pi]$.

The Wilson line was considered in [6] for the case of $\mathcal{N}=2^*$ obtaining the following field theory prediction at strong coupling

$$\ln \langle W(C)\rangle_{\text{line}} = \mathrm{ML}\left[\frac{\sqrt{\lambda}}{2\pi} - \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right], \tag{3.10}$$

where M is the mass parameter and L is the length of C. In [46] the first term on the r.h.s. was successfully matched with the classical string partition function of the Pilch-Warner background. Obtaining the λ^0 term in string theory is the main result of paper I. In the process, we also reproduced the result (3.8) in a simple and elegant manner. The classical string solution for the Pilch-Warner background has naturally the same geometry and is parametrized by $c_{\text{cl}}=\sigma$ and $x^0_{\text{cl}}=\tau$ in the coordinates of section 2.3.

3.4.2 The circular Wilson loop

The circular Wilson loop operator is 1/2 BPS and its parametrization is given by

$$x^\mu(s) = (\cos s, \sin s, 0, 0), \qquad\qquad n^i(s) = (0,0,1,0,0,0).$$

In the field theory side, in $\mathcal{N}=4$ SYM the expectation value for this operator was conjectured to be described by a Gaussian matrix model in [8, 23] and its result was proven using supersymmetric localization in [2]. The expectation value for this Wilson loop is known for all values of the rank of the gauge group (N) and all values of λ. In the planar limit the result is

$$\langle W(C)\rangle_{\text{circle}} = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right), \tag{3.11}$$

which has the following behaviour at strong coupling

$$\ln \langle W(C)\rangle_{\text{circle}} = \sqrt{\lambda} - \frac{3}{4}\ln\lambda - \frac{1}{2}\ln\frac{\pi}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{3.12}$$

In the string theory side, this Wilson loop is described by the following classical solution describing a circle in the boundary of AdS

$$x^\mu_{\text{cl}} = \left(\frac{\cos\tau}{\cosh\sigma}, \frac{\sin\tau}{\cosh\sigma}, 0, 0\right), \qquad z_{\text{cl}} = \tanh\sigma, \qquad \psi_{\text{cl}} = 0, \qquad \varphi_{\text{cl}} = \text{any},$$

while the angles ϕ_i with $i\in\{1,2,3\}$ are arbitrary constants. In the expression above $\sigma\in[0,\infty)$ and $\tau\in[0,2\pi]$.

The string theory computation of (3.12) in $AdS_5\times S^5$ has only been successful at leading order in λ. The first term in equation (3.12) is reproduced by considering the string action evaluated at the classical solution. The $\ln\lambda$ term has been conjectured to be produced by the normalization of ghost zero-modes in the partition function. Meanwhile, the remaining term in (3.12) is understood to come from fluctuations around the classical solution. Despite several attempts [7, 9, 10], exact matching of (3.12) with string theory calculations has not been achieved for the last term, while the $\ln \lambda$ term has not been properly explained.

This Wilson loop can also be studied in the case when it has a winding k, in which case the equations (3.11) and (3.12) are valid after the substitution $\lambda \to k^2\lambda$. The dependence on k is also a mystery from the string theory side as calculations based on Gel'fand-Yaglom [9] and heat kernel methods [11, 14] have lead to discrepancies with localization results.

3.4.3 The latitude Wilson loop

The latitude Wilson loop operator describes a family of Wilson loops parametrized by an angle θ_0 describing a latitude in a $S^2 \subset S^5$ and finishing in a circle at the boundary of AdS. The operator is 1/4 BPS and is parametrized by

$$x^\mu(s) = (\cos s, \sin s, 0, 0)\,, \quad n^i(s) = (\sin\theta_0 \cos s, \sin\theta_0 \sin s, \cos\theta_0, 0, 0, 0)\,.$$

It is easy to see that the above Wilson loop in $\mathcal{N} = 4$ SYM reduces to the 1/2 BPS circular Wilson loop described in section 3.4.2 when $\theta_0 = 0$, while for $\theta_0 = \pi/2$ it corresponds to the Wilson loop studied in [47]. The predictions from localization for its expectation value are obtained through the substitution $\lambda \to \lambda\, \cos^2\theta_0$ resulting in [48]

$$\langle W(C)\rangle_{\text{latitude}} = \frac{2}{\sqrt{\lambda}\cos\theta_0} I_1\left(\sqrt{\lambda}\cos\theta_0\right)$$

in the large N limit and

$$\ln\langle W(C)\rangle_{\text{latitude}} = \sqrt{\lambda} - \frac{3}{4}\ln\lambda - \frac{3}{2}\ln\cos\theta_0 - \frac{1}{2}\ln\frac{\pi}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

at strong coupling.

The corresponding classical string solution describes a circle in the boundary of AdS and extends into an $S^2 \subset S^5$

$$\begin{aligned} x^\mu_{\text{cl}} &= \left(\frac{\cos\tau}{\cosh\sigma}, \frac{\sin\tau}{\cosh\sigma}, 0, 0\right), \qquad & z_{\text{cl}} &= \tanh\sigma, \\ \cos\psi_{\text{cl}} &= \tanh(\sigma + \sigma_0)\,, & \varphi_{\text{cl}} &= \tau, \end{aligned} \tag{3.13}$$

where $\tau \in [0, 2\pi]$, $\{\sigma, \sigma_0\} \in [0, \infty)$ and σ_0 is written in terms of θ_0 as

$$\tanh\sigma_0 = \cos\theta_0.$$

The string theory computation of this Wilson loop has not been considered independently as the particular case $\theta_0 = 0$ is already problematic. Instead, efforts have been done on studying the ratio between the semiclassical partition functions of a latitude with arbitrary angle θ_0 and the case $\theta_0 = 0$ (which was described in section 3.4.2). The hope is that since both worldsheets have the same topology, measure factors and possible ghost contributions would cancel.

This ratio of Wilson loops is understood at leading order in λ in terms of the classical actions [48]. However, computations of the semiclassical string partition function in $AdS_5 \times S^5$ led to discrepancies with the localization result when the determinants were evaluated using the Gel'fand-Yaglom method [12, 13]. Agreement was reached at first order in perturbation theory for very small θ_0 using a series expansion of the heat kernel [14], and more recently using zeta function regularization [15]. In paper II perfect agreement was reached with the localization prediction for this ratio at all orders in θ_0.

4. Functional determinants

As we discussed in the previous chapter, in the framework of the gauge-string duality, the string theory computation of Wilson loops consists in the evaluation of a string partition function around a classical string configuration. At first order at strong coupling, contributions come from the minimal area described by the string worldsheet. Meanwhile, computation of 1-loop corrections to the string partition function reduces to the problem of evaluating determinants of several differential operators. The later is a highly non-trivial task as determinants of differential operators themselves are usually divergent quantities.

Evaluation of determinants of differential operators plays an important role in theoretical physics, as it is a technique appearing extensively in many areas of physics and mathematics. Motivation to study such techniques in physics comes for example from the study of effective actions appearing in relativistic and non-relativistic many-body physics, tunnelling and nucleation processes [49, 50], lattice gauge theories, gauge fixing and Faddeev-Popov determinants, evaluation of entanglement entropy, etc. Despite the widespread appearance of functional determinants in mathematical physics, exact results are only known for few cases like the Klein-Gordon or Dirac operators on spheres or tori. For differential operators on arbitrary backgrounds the picture is not so clear and at times one has to rely heavily on approximation methods.

Historically, the recurrent appearance of differential operators in theoretical physics came partly due to the appearance of Quantum Mechanics [42]. Later, it became known that the spectrum of such operators can be described in terms of spectral functions, the most commonly used being the heat kernel. This method played an important role in the understanding of quantum corrections due to the work of DeWitt in the 1960's [43]. Later, it was realized that information on the spectrum of differential operators could be encompassed in terms of the zeta function of the operator. More sophisticated techniques like the Gel'fand-Yaglom method [51] have received widespread use, as they allow for the computation of functional determinants without requiring full knowledge of the eigenfunctions in question.

In this chapter we will briefly discuss the main features of some of these methods in section 4.1. In section 4.2, based on the previous section, we review some results for the contribution of conformal factors to functional determinants relevant for the calculation in paper II. Section 4.3 presents identities which will simplify the calculation in paper I. Then in section 4.4 we motivate the method based on the scattering phaseshifts, which will be used later for the calculation of 1-loop string corrections of Wilson loops in papers I and II.

4.1 Zeta function & heat kernel

Due to the large amount of literature on the subject, we only present the main features of these techniques which will be useful later in section 4.2. Heat kernel and zeta function techniques will not be explicitly used in this thesis for the evaluation of determinants. However, identities from these approaches will be useful in the calculation of paper II for a check on the underlying assumptions. The presentation of the topic in this section is largely based on [42, 43] with many details deliberately swept under the rug.

Let $\mathcal{K}$ be a self-adjoint second order differential operator, its determinant can be written in the form

$$\ln \det \mathcal{K} = -\zeta' (0; \mathcal{K}) , \tag{4.1}$$

where $\zeta(s; \mathcal{K})$ is the zeta function of $\mathcal{K}$. The later is usually defined in terms of the eigenvalues λ of $\mathcal{K}$ through

$$\zeta (s; \mathcal{K}) = \sum_{\lambda} \lambda^{-s}, \tag{4.2}$$

where we assume that $\lambda \in \mathbb{R}$ and $\lambda > 0$.

From the equation above, explicit knowledge of all eigenvalues λ would in principle lead to the calculation of the determinant. However, it is not immediate that this series converges as for large values of λ the r.h.s. of (4.2) converges if Re $s > n/2$ where n denotes the dimensionality of the manifold ($n = 2$ in our case as the string worldsheet is 2 dimensional). Analytic extension of the zeta function can be done for values of s in the rest of the complex plane. Moreover, this expression can also be extended to operators with negative eigenvalues and Dirac type operators. These extensions will not be used in our work as our calculations do not rely on the zeta function method, however zeta function methods have recently been used in the study of latitude and circular Wilson loops in [15] and [52].

One of the main advantages of the zeta function method is its simplicity and the fact that it is easily connected to other methods for evaluation of determinants, in particular the heat kernel method. The later is a widely used method based on knowledge of a function $K(x, y|t)$ satisfying

$$(\partial_t + \mathcal{K}_x) K (x, y|t) = 0, \qquad K (x, y|0) = \delta^{(n)} (x - y) , \tag{4.3}$$

where $K(x, y|t)$ is usually referred to as the "heat kernel". Despite the relative simplicity of (4.3), the heat kernel is only known for few Laplace and Dirac operators in very symmetric manifolds like the sphere or flat-space. For more complicated geometries, like that of the Pilch-Warner or the latitude Wilson loop, the corresponding heat kernel is not known.

The heat kernel of the operator $\mathcal{K}$ can be written in the form

$$K (x, y|t) = \langle x| e^{-t\mathcal{K}} |y\rangle , \tag{4.4}$$

and tracing over x and y leads to

$$\begin{aligned} K(\mathcal{K};t) = K(1,\mathcal{K};t) &= \int \sqrt{h}\; K(x,x|t)\, dx^n \\ &= \sum_\lambda e^{-t\lambda}, \end{aligned} \tag{4.5}$$

where in the first line we used the definition of the "heat trace"

$$K(f,\mathcal{K};t) = \mathrm{Tr}\left[f\; e^{-t\mathcal{K}}\right] \tag{4.6}$$

evaluated for a "smoothing function" $f = 1$. Meanwhile, the second line of (4.5) is a consequence of taking the trace of (4.4).

The bridge between the heat kernel and zeta function methods is made by the equation

$$\zeta(s;\mathcal{K}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(\mathcal{K};t)\; dt, \tag{4.7}$$

which is a direct consequence of (4.2) and the second line of (4.5). Equation (4.7) connects the heat trace of the operator $\mathcal{K}$ with its zeta function, making it possible to calculate $\det \mathcal{K}$ provided one knows the heat kernel $K(x,y|t)$. The determinant of $\mathcal{K}$ can be written in terms of the heat trace as

$$\ln\det\mathcal{K} = -\lim_{s\to 0} \frac{d}{ds}\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\mathrm{Tr}\; e^{-t\mathcal{K}} dt. \tag{4.8}$$

We conclude our presentation of the heat kernel method by introducing the "heat kernel coefficients" which are the coefficients a_p in a series expansion of the heat trace for $t \to 0$

$$K(f,\mathcal{K};t) = \sum_{p=0}^\infty a_p(f,\mathcal{K})\, t^{(p-n)/2}. \tag{4.9}$$

The heat kernel coefficients are very useful as they depend on information on the operator, the manifold and the boundary conditions, and can provide information on the functional determinants as will be shown in section 4.2. Depending on the imposed boundary conditions, an expression for the heat coefficients may or may not be known in the literature. For later use, we present the coefficient $a_2(f,\mathcal{K})$ for Dirichlet boundary conditions

$$\begin{aligned} a_2(f,\mathcal{K}) = \frac{1}{(4\pi)^{n/2}} \Bigg[&\int_{\mathcal{M}} \sqrt{h}\; \mathrm{Tr}\left(f\left(\frac{R}{6} - E\right)\right) dx^n \\ &+ \int_{\partial\mathcal{M}} \sqrt{g}\; \mathrm{Tr}\left(\frac{f}{3}K^j_j - \frac{1}{2}\nabla_n f\right) dx^{n-1}\Bigg], \end{aligned} \tag{4.10}$$

where K^i_j is the extrinsic curvature at the boundary, g_{ij} is the metric at the boundary, ∇_n the covariant derivative projected along the direction normal to the boundary and E comes from writing the operator in "covariant form"

$$\mathcal{K} = -h^{ij} D_i D_j + E, \tag{4.11}$$

where D_i can have Riemann and gauge connections parts. When the operator of interest is linear in derivatives, as is usually the case for fermions, one considers the square of the operator.

Several identities presented in this section for the heat kernel will be useful for a check on the assumptions of paper II. This check is based on the machinery developed in the next section.

Before finishing our discussion on the main features of the heat kernel method, we discuss their role in string theory Wilson loop computations. Despite the fact that the heat kernel method can not be used for the straight line computation of paper I due to ignorance on the heat kernel for the Pilch-Warner geometry, heat kernel computations of the straight line in $AdS_5 \times S^5$ have successfully reproduced the field theory prediction [7].

The heat kernels needed for the 1/2 BPS circular Wilson loop computation are known [53, 54, 55, 56], but there is a mismatch with the field theory prediction [7]. For the case of the latitude Wilson loop studied in paper II, no expression is know for the heat kernel, however, a perturbative result starting from the known heat kernel of the case $\theta_0 = 0$ led to reproducing the localization result at first order in perturbation theory of the ratio of circular and latitude loops [14].

Using the heat kernel of the circular Wilson loop and the Sommerfeld formula, the case of the k-winding circular Wilson loop was studied in [11], leading to a mismatch with field theory predictions. In spite of the vast mathematical literature on this method, its applications for the string theory calculation of Wilson loops seem rather behind.

4.2 Determinants & conformal factors

The operators considered in paper II have a factor depending on σ in front of the ∂_τ^2 and ∂_σ^2 derivatives. The method we use to evaluate determinants, which will be introduced in section 4.4, requires this factor to be -1. The dependence of determinants in terms of these factors is given through the Seeley coefficients introduced in the previous section. We will first show how the dependence on the conformal factors is encoded in the heat kernel coefficients and then we will derive concrete expressions for bosonic and fermionic operators.

The bosonic operators of interest can all be written in the canonical form

$$\mathcal{K}(\alpha) = \mathrm{e}^{2\alpha\phi} \left(-\delta^{ij} D_i D_j + E \right), \tag{4.12}$$

where $\alpha \in [0, 1]$ is a constant, ϕ is a scalar function depending on the worldsheet coordinates, the derivative can have a "gauge" part $D_j = \partial_j + iA_j$, while A_j and E are $n \times n$ matrices which can depend on both τ and σ.

On the other hand, the fermionic operators of interest in paper II are 2×2 Dirac operators of the form

$$\mathcal{D}(\alpha) = \mathrm{e}^{\frac{3\alpha\phi}{2}} \left(i\gamma^j D_j + \gamma^3 a + \mathbb{1} v \right) \mathrm{e}^{-\frac{\alpha\phi}{2}}, \tag{4.13}$$

where the contraction is done with the Euclidean metric, a and v are scalar functions depending on the worldsheet coordinates, while $D_j = \partial_j + iA_j$ with A_j a 2×2 matrix that can depend on $\{\tau, \sigma\}$ and is proportional to the identity matrix. Just as for bosons $\alpha \in [0, 1]$ is a constant parameter and ϕ is a scalar function depending on worldsheet coordinates. To make direct comparison with the fermionic operators of paper II, it is convenient to use the following basis

$$\gamma^\tau = -\tau_2, \qquad \gamma^\sigma = \tau_1, \qquad \gamma^3 \equiv -i\varepsilon_{\mu\nu}\gamma^\mu\gamma^\nu/2 = \tau_3.$$

We are interested in how the determinants of the operators (4.12) and (4.13) change when the parameter α takes the values $\alpha = 0$ and $\alpha = 1$. When $\alpha = 1$ we will recover the operators obtained from the Green-Schwarz action, while $\alpha = 0$ produces the operators for which we will calculate the determinants with the phaseshift method. The dependence of the determinants on α can be explicitly shown using expressions from section 4.1 and following the arguments in [3].

From equation (4.12) it is easy to see that

$$\frac{\partial}{\partial\alpha} \operatorname{Tr} \mathrm{e}^{-t\mathcal{K}} = 2t \frac{\partial}{\partial t} \operatorname{Tr} \phi \, \mathrm{e}^{-t\mathcal{K}}. \tag{4.14}$$

Differentiating (4.8) with respect to α and replacing the expression above results in

$$\frac{d}{d\alpha} \ln\det\mathcal{K} = 2 \lim_{s\to 0} \frac{d}{ds} \frac{s}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \operatorname{Tr} \phi \, \mathrm{e}^{-t\mathcal{K}}. \tag{4.15}$$

The integrand above can be badly behaved around $t = 0$. To perform this integral we use the expressions (4.6) and (4.9) for the heat trace, as well as the identity

$$\int_0^\infty dt \, t^s f(t) \overset{s\to 0}{=} \frac{1}{s} \operatorname*{res}_{t=0} f(t) + \text{regular}.$$

Doing so results in

$$\frac{d}{d\alpha} \ln\det\mathcal{K} = 2a_2(\phi|\mathcal{K}). \tag{4.16}$$

In summary, the dependence of functional determinants on the conformal factor is through the DeWitt-Seeley coefficient a_2.

Using equations (4.10) and (4.11), it can be shown that the DeWitt-Seeley coefficient for the bosonic operator defined in (4.12) is

$$a_2(\phi, \mathcal{K}) = \frac{1}{4\pi} \int d\sigma^2 \left(\frac{\alpha n}{3} \phi \delta^{ij} \partial_i \partial_j \phi - \phi \operatorname{Tr} E \right) + \frac{n}{24\pi} \oint ds \, (2\alpha\phi\partial_n\phi - 3\partial_n\phi).$$

Integrating with respect to α we finally obtain

$$\ln \frac{\det\mathcal{K}(1)}{\det\mathcal{K}(0)} = \frac{1}{2\pi} \int d^2\sigma \left(\frac{n}{6} \phi\delta^{ij}\partial_i\partial_j\phi - \phi \operatorname{Tr} E \right) + \frac{n}{12\pi} \oint ds \, (\phi\partial_n\phi - 3\partial_n\phi) \, . \tag{4.17}$$

To compute how the determinant of the fermionic operator in (4.13) depends on the conformal factor, we must first bring it to the canonical form (4.11) by squaring, which results in

$$\begin{aligned}\mathcal{D}^2(\alpha) &= \mathrm{e}^{\,2\alpha\phi}\Big[-\nabla_i\nabla^i + \frac{\alpha}{2}\,\partial_i\partial^i\phi + a^2 - v^2 \\ &\quad + \varepsilon^{ij}\left(\partial_i a + \alpha a\partial_i\phi\right)\gamma_j + \left(\frac{1}{2}\,\varepsilon^{ij}F_{ij} + 2av\right)\gamma^3\Big], \end{aligned} \tag{4.18}$$

where contractions are done with the Euclidean metric δ_{ij} and

$$\nabla_j = D_j - iv\gamma_j - \frac{i\alpha}{2}\,\varepsilon^{jk}\partial_k\phi\gamma^3. \tag{4.19}$$

Following the same arguments used for bosons, it can be shown that for the fermionic operator

$$\frac{d}{d\alpha}\,\ln\det\mathcal{D}^2 = 2a_2(\phi|\mathcal{D}^2). \tag{4.20}$$

By explicit computation using equations (4.10) and (4.18), the DeWitt-Seeley coefficient for the fermionic operator results in

$$\begin{aligned}a_2\left(\phi,\mathcal{D}^2\right) = -\frac{1}{2\pi}\int d\sigma^2\left(\frac{\alpha}{6}\phi\delta^{ij}\partial_i\partial_j\phi + \phi\left(a^2 - v^2\right)\right) \\ + \frac{1}{12\pi}\oint ds\,(2\alpha\phi\partial_n\phi - 3\partial_n\phi).\end{aligned}$$

Integration over α results in the final expression for fermions

$$\begin{aligned}\frac{1}{2}\,\ln\frac{\det\mathcal{D}^2(0)}{\det\mathcal{D}^2(1)} &= \frac{1}{2\pi}\int d^2\sigma\,\left[\frac{1}{12}\,\phi\delta^{ij}\partial_i\partial_j\phi + \phi\left(a^2 - v^2\right)\right] \\ &\quad - \frac{1}{12\pi}\oint ds\,\left(\phi\partial_n\phi - 3\partial_n\phi\right).\end{aligned} \tag{4.21}$$

Equations (4.17) and (4.21) will play an important role in the computation of paper II as they give us information on how the determinants are affected by this conformal factor, which needs to be removed in order to evaluate the determinants.

4.3 Isospectral operators

Depending on the operators being studied, it is sometimes possible to make statements on the spectrum of two operators without having to explicitly calculate all the set of eigenvectors and eigenvalues. Take for instance the second order differential operators given by

$$\mathcal{K}_{\mathrm{I}} = L^\dagger L, \qquad\qquad \mathcal{K}_{\mathrm{II}} = LL^\dagger, \tag{4.22}$$

where L and $L^\dagger$ denote operators linear in derivatives. By simple algebra it is easy to see that

$$\mathcal{K}_{\mathrm{I}}L^\dagger = L^\dagger\mathcal{K}_{\mathrm{II}}, \qquad\qquad \mathcal{K}_{\mathrm{II}}L = L\mathcal{K}_{\mathrm{I}}.$$

These relations, also called "intertwining relations" allow to map the spectrum of one operator into another. Let us consider $\psi_{\rm I}$ and $\psi_{\rm II}$ eigenfunctions of $\mathcal{K}_{\rm I}$ and $\mathcal{K}_{\rm II}$ satisfying

$$\mathcal{K}_{\rm I}\psi_{\rm I} = \Lambda_{\rm I}\psi_{\rm I}, \qquad \mathcal{K}_{\rm II}\psi_{\rm II} = \Lambda_{\rm II}\psi_{\rm II}. \tag{4.23}$$

It is easy to see to check that

$$\begin{aligned} \mathcal{K}_{\rm I}\left(L^\dagger\psi_{\rm II}\right) &= \left(\mathcal{K}_{\rm I}L^\dagger\right)\psi_{\rm II} = L^\dagger\mathcal{K}_{\rm II}\psi_{\rm II} = \Lambda_{\rm II}\left(L^\dagger\psi_{\rm II}\right), \\ \mathcal{K}_{\rm II}\left(L\psi_{\rm I}\right) &= \left(\mathcal{K}_{\rm II}L\right)\psi_{\rm I} = \left(L\mathcal{K}_{\rm I}\right)\psi_{\rm I} = \Lambda_{\rm I}\left(L\psi_{\rm I}\right). \end{aligned}$$

Thus, $L^\dagger\psi_{\rm II}$ is an eigenvalue of $\mathcal{K}_{\rm I}$ and $L\psi_{\rm I}$ is an eigenvalue of $\mathcal{K}_{\rm II}$. In this way, one can map the eigenvalues and eigenvectors of one operator into the other. Operators of the type (4.22) are commonly called "isospectral" operators as the spectrum of the two can be identified up to zero modes.

In principle, when considering the evaluation of determinants it is not sufficient for the operators to be isospectral in order for their determinants to be the same. To see this more clearly one may think of the eigenfunctions $\psi_{\rm I}$ and $\psi_{\rm II}$ as each being a superposition of two solutions to the spectral problems in (4.23). In order for the eigenvalues of $\psi_{\rm I}$ and $\psi_{\rm II}$ to contribute to the determinants of $\mathcal{K}_{\rm I}$ and $\mathcal{K}_{\rm II}$, the eigenfunctions must satisfy the boundary conditions of the problem. Imposing boundary conditions fixes the coefficients in the superpositions of $\psi_{\rm I}$ and $\psi_{\rm II}$. If the choice of boundary conditions is compatible with the map

$$\psi_{\rm I} \propto L^\dagger\psi_{\rm II}, \qquad \psi_{\rm II} \propto L\psi_{\rm I}.$$

then it is possible to identify the eigenfunctions contributing to the two determinants and the later will be the same.

Operators of this type are commonly seen in supersymmetric quantum mechanics. Due to the underlying supersymmetry of string theory it is not entirely surprising to find that some of the operators appearing in semiclassical string partition functions share this property. The later is the case for a fraction of the operators appearing in the computation in paper I which will simplify the calculations considerably.

4.4 Phaseshifts & determinants

The method we will use to evaluate the functional determinants in papers I and II relies on explicit calculation of the eigenfunctions and eigenvalues of the differential operators. The operators we are interested in can be reduced after Fourier expansion in the τ coordinate into 1-dimensional operators of the form

$$\mathcal{K} = -\partial_\sigma^2 + V(\sigma)\,, \tag{4.24}$$

with the following asymptotic behaviour

$$\mathcal{K}_\infty = \lim_{\sigma\to\infty}\mathcal{K} = -\partial_\sigma^2 + V_\infty, \tag{4.25}$$

where $V_\infty = V(\infty)$ is a constant.

The spectrum of this type of operators can be qualitatively divided in three parts. The first one consist on a continuum of exponentially increasing/decreasing functions. The second is a continuos spectrum of oscillating functions. The third is composed by a finite number of discrete bound states. Naturally, to evaluate functional determinants it is necessary to choose a set of boundary conditions and an inner product. In principle, the determinant must only receive contributions of eigenfunctions that satisfy both the boundary conditions and that have a finite norm.
In flat space, exponentially increasing functions, and in most cases bound states, will not have a finite norm. Meanwhile, the boundary conditions we are interested in consist of Dirichlet boundary conditions at the origin[1]

$$\psi(\sigma = 0) = 0. \tag{4.26}$$

This type of boundary condition excludes exponentially decreasing functions and some of the discrete parts of the spectrum. Thus, we are left with oscillating functions and a finite number of discrete eigenfunctions contributing to the determinant.

As discussed previously, evaluation of determinants of elliptic differential operators usually results in divergences. The reason for this is that the determinant in question consists of an infinite product of eigenvalues. Consequently, one needs a regularization prescription. One option, as for instance done in the zeta function formalism of section 4.1, consists in removing the divergent pieces in such a way that only the physically meaningful finite piece remains. Another possibility, which is the one we will use, consists on considering the ratio of two determinants instead of evaluating individual determinants. The justification for this resides in the fact that in many cases the divergent pieces cancel each other, obtaining a finite result.

An ideal differential operator to use as a regulator in the ratio is the corresponding asymptotic operator $\mathcal{K}_\infty$. Given the asymptotic differential operator (4.25), it is easy to see that the spectral problem

$$\mathcal{K}_\infty \psi_\infty = \left(-\partial_\sigma^2 + V_\infty\right) \psi_\infty = E_\infty(p)\ \psi_\infty$$

has for solutions plane waves of wave number p. After imposing the boundary condition at the origin, the asymptotic eigenfunctions are of the form

$$\psi_\infty \propto \sin\left(p\sigma\right),$$

with eigenvalue

$$E_\infty\left(p\right) = p^2 + V_\infty.$$

Meanwhile, the original spectral problem

$$\mathcal{K}\ \psi = \left(-\partial_\sigma^2 + V\left(\sigma\right)\right) \psi = E(p)\ \psi \tag{4.27}$$

[1] In paper I, the coordinates are such that $\sigma \in [1, \infty)$. Consequently, the boundary condition used is $\psi(\sigma = 1) = 0$.

can be seen as a Schrödinger problem with potential $V(\sigma)$. Depending on the potential $V(\sigma)$, analytic solutions for $\psi(\sigma)$ may be found as in paper II or one may need to resort to numerics as in paper I. In any case, since $\mathcal{K}$ limits to $\mathcal{K}_\infty$ for large σ, the eigenfunctions $\psi(\sigma)$ will behave as plane waves asymptotically

$$\lim_{\sigma\to\infty} \psi(\sigma) \propto \sin\left(p\sigma + \delta(p)\right),$$

where $\delta(p)$ is a phaseshift generated by the potential.

Physically, one can think of the continuum spectrum of $\mathcal{K}$ as consisting of plane waves coming from infinity and interacting with the potential close to the origin. This scattering of the incident wave with the potential will produce a shift depending on the momentum p of the incident wave.

To evaluate the determinant we put the system in a box, which introduces an IR regulator R that corresponds to the length of the box. At a very large $\sigma = R$ we impose the quantization condition on the scattering states

$$pR + \delta(p) = \pi n,$$

which basically amounts to $\psi(R) = 0$. This quantization condition implies a density of states

$$\rho(p) = \frac{dn}{dp} = \frac{\delta'(p)}{\pi} + \frac{R}{\pi}.$$

Meanwhile, for the asymptotic operator, for which there is no scattering, the quantization condition is

$$pR = \pi n, \qquad \rho_\infty(p) = \frac{dn}{dp} = \frac{R}{\pi}, \tag{4.28}$$

due to the absence of the potential.

In terms of the densities of states we have that

$$\ln \frac{\det \mathcal{K}}{\det \mathcal{K}_\infty} = \sum_n E_n^{\mathrm{BS}} + \int_0^\infty \ln\left(p^2 + V_\infty\right) \frac{\delta'(p)}{\pi} dp, \tag{4.29}$$

where the sum is over the discrete eigenvalues of $\mathcal{K}$ and the integration is over the ratio of continuum of states. This equation will be used later on for evaluating the different determinants considered in papers I and II.

As we mentioned earlier, the spectral problems we will consider can be reduced to one dimension by Fourier expanding in the worldsheet direction τ. Consequently, for the problems of interest the r.h.s. of (4.29) will have summation/integration over Fourier frequencies.

Naturally, since we will consider the superstring, operators of Dirac type will enter in the calculations of papers I and II. Operators of this type are not of the form (4.24), but we can square its eigenvalues and use (4.29). The reason why this works is that for the 2×2 Dirac operators considered, decoupling of the components leads to spectral problems of the type (4.27) for each component. Since the components are coupled, it is not always possible to impose (4.26) to both components. Instead, we will impose this condition to only one of the components picking the better behaved solution at the origin. We will discuss this in more detail when doing the calculations of chapters 5 and 6.

5. Wilson line in the Pilch-Warner background

In this section we compute the semiclassical string partition function for the straight line in the Pilch-Warner background which is the main result of paper I. As we discussed in section 3.4, this configuration is dual to the Wilson line of $\mathcal{N} = 2^*$ and its prediction from localization at strong coupling is

$$\ln \langle W(C) \rangle_{\text{line}} = \text{ML} \left[\frac{\sqrt{\lambda}}{2\pi} - \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}} \right) \right]. \tag{5.1}$$

Here we calculate this quantity from the string side of the gauge-string correspondence based on the concepts introduced in chapters 2, 3 and the techniques presented in chapter 4. For convenience we set M = 1, dependence on this variable can be easily recovered by dimensional analysis.

The string theory dual to the Wilson line in $\mathcal{N} = 2^*$ is given by the classical string solution

$$x^1_{\text{cl}} = \tau, \qquad\qquad c_{\text{cl}} = \sigma, \tag{5.2}$$

with $\tau \in [-\text{L}/2, \text{L}/2]$ and $\sigma \in [1, \infty)$, while all other fields are set to zero. From the gauge fixing procedure discussed in section 3.3, the worldsheet metric takes the value of the induced metric (3.4) resulting in

$$ds^2_{\text{w.s.}} = \frac{A}{\sigma^2 - 1} d\tau^2 + \frac{1}{A(\sigma^2 - 1)^2} d\sigma^2, \tag{5.3}$$

where now $A \equiv A(\sigma)$.

Using the field content presented in section 2.3, we proceed in sections 5.1 and 5.2 to compute the classical and semiclassical pieces entering equation (3.7). Later, in section 5.3 we discuss several aspects concerning the evaluation of the functional determinants. Finally, in section 5.4 we discuss the end result, as well as the numerical setup needed for its calculation. To keep the presentation brief several technical details are omitted here, the reader interested in technicalities is referred to the original work in paper I.

5.1 The classical partition function

Computation of the contribution coming from the classical partition function follows from our discussion in section 3.3.1

$$S_B(X_{\text{cl}}) = \frac{\sqrt{\lambda}}{2\pi} \int\limits_{\text{reg}} \frac{d\tau d\sigma}{(\sigma^2 - 1)^{3/2}} = -\frac{\sqrt{\lambda}}{2\pi} \text{L}, \tag{5.4}$$

where integration over τ was performed in the interval $-\mathrm{L}/2$ to $\mathrm{L}/2$, while for σ the interval was from $1+\epsilon^2/2$ to infinity. The later is a consequence of the coordinate c from Pilch-Warner being related to the holographic coordinate of AdS by $c \to 1 + z^2/2$ for small z and the cutoff ϵ being located at $z = \epsilon$. As discussed in section 3.3.1, we use the regularization prescription of removing the $1/\epsilon$ divergences in the final result. From combining (3.7) and (5.4), we see that the result above reproduces the localization prediction (5.1) at order $\lambda^{1/2}$.

Since the Pilch-Warner background has a non-trivial dilaton, there is a contribution from the Fradkin-Tseytlin term presented in equations (2.2) and (2.3). Evaluation of this term at the classical solution results in

$$\mathrm{S}_{\mathrm{FT}}\left(X_{\mathrm{cl}}\right) = \frac{\mathrm{L}}{2\pi}\int_1^\infty \frac{d\sigma}{\sigma\sqrt{\sigma^2-1}} = \frac{\mathrm{L}}{4}. \tag{5.5}$$

Comparison of this result with the localization prediction (5.1) shows that this term corresponds to half of the λ^0 final result. The other half results from the evaluation of quantum fluctuations around the classical solution and their respective determinants.

Writing both contributions together, we obtain the following result for the classical piece of the partition function

$$S_{\mathrm{String}}\left(X_{\mathrm{cl}}\right) = -\mathrm{L}\left(\frac{\sqrt{\lambda}}{2\pi} - \frac{1}{4}\right). \tag{5.6}$$

5.2 The semiclassical partition function

We now proceed to compute the contributions coming from the semiclassical string partition function, just as explained in section 3.3.2. First, in sections 5.2.1 and 5.2.2 we present the bosonic and fermionic operators obtained from the Green-Schwarz action by expanding up to second order in fluctuations. Then, in section 5.2.3 we collect all contributions and use several identities to simplify the computation.

5.2.1 Bosonic operators

Expanding the bosonic action to second order in fluctuations and projecting in the tangent space (recall $\delta X^\mu = E^\mu_{\hat{a}}\xi^{\hat{a}}$), we obtain a structure of the form

$$S_{\mathrm{B}} = \sum_a \int d\tau d\sigma\sqrt{h}\;\xi^{\hat{a}}\mathcal{K}_a\xi^{\hat{a}},$$

where summation is over the eight coordinates transversal to the worldsheet and the operators $\mathcal{K}_a$ are given by

$$
\begin{aligned}
\mathcal{K}_{\mathbf{x}} &= -\partial_\tau^2 - A^2(\sigma^2-1)\partial_\sigma^2 + A\,(4-3A\sigma)\,\partial_\sigma\,, && (5.7)\\
\mathcal{K}_{\phi} &= \mathcal{K}_{\mathbf{x}} - \frac{2A\sigma}{\sigma^2-1}, && (5.8)\\
\mathcal{K}_{\mathbf{y}}^{\pm} &= \mathcal{K}_{\mathbf{x}} + 1 - \frac{A\left(\sigma^2+1\right)\left[4\sigma+3A(\sigma^2-1)\right]}{4\sigma^2(\sigma^2-1)} \pm \frac{A}{\sigma}\,\partial_\tau\,, && (5.9)
\end{aligned}
$$

where $\mathcal{K}_{\mathbf{x}}$ corresponds to fluctuations along the three AdS coordinates transversal to the worldsheet, while $\{\mathcal{K}_\phi, \mathcal{K}_{\mathbf{y}}^+, \mathcal{K}_{\mathbf{y}}^-\}$ are three operators corresponding to fluctuations in the deformed five-sphere with each having multiplicity 1, 2 and 2, respectively.

In the expansion of the bosonic action, the operators $\mathcal{K}_{\mathbf{y}}^+$ and $\mathcal{K}_{\mathbf{y}}^-$ originally appeared coupled as a 2×2 second order differential operator, but through a similarity transformation the later was brought to a block diagonal form with $\mathcal{K}_{\mathbf{y}}^{\pm}$ in the diagonal components.

In order for the operators above to have the structure $\mathcal{K} = -\partial_\tau^2 + ...$, we also performed a rescaling of fluctuations of the form

$$\xi^{\hat a} \to \sqrt{\frac{A}{\sigma^2-1}}\,\xi^{\hat a}$$

and used partial integration. This rescaling of fluctuations will be compensated by a similar rescaling for fermionic operators, thus preserving the measure of the path integral.

In terms of these operators, the bosonic contribution to the semiclassical partition function (3.7) is given by

$$\det \mathcal{K}_B = \det{}^3\mathcal{K}_x \det \mathcal{K}_\phi \det{}^2\mathcal{K}_y^+ \det{}^2\mathcal{K}_y^-\,. \qquad (5.10)$$

5.2.2 Fermionic operators

The computation of second order fluctuations for the fermionic case amounts to evaluating equation (2.4) at the classical solution. Using the orthonormal frame

$$
\begin{array}{lllll}
E^{\hat 0} \propto dx^0, & E^{\hat 1} \propto dx^1, & E^{\hat 2} \propto dx^2, & E^{\hat 3} \propto dx^3, & E^{\hat 4} \propto dc,\\
E^{\hat 5} \propto d\theta, & E^{\hat 6} \propto \sigma_1, & E^{\hat 7} \propto \sigma_2, & E^{\hat 8} \propto \sigma_3, & E^{\hat 9} \propto d\phi,
\end{array}
$$

the basis for Dirac matrices presented in [57], the field content of the Pilch-Warner background and the κ symmetry gauge fixing discussed in section 3.3, we obtain the following result

$$
\begin{aligned}
L_{\mathrm{F}}^{(2)} =& 2\sqrt{h}\,\bar\chi\,\Big[\sqrt{c_{(1)}}\gamma^{\hat 1}\partial_\tau + \sqrt{c_{(2)}}\gamma^{\hat 4}\partial_\sigma + c_{(\omega)}\gamma^{\hat 4} - c_{(5)}^{\mathrm{RR}}\gamma^{\hat 1\hat 4}\\
&-c_{(1)}^{\mathrm{RR}}\gamma^{\hat 1\hat 4\hat 9} - i c_{(3)}^{\mathrm{NSNS}}\left(\gamma^{\hat 1\hat 5\hat 6} - \gamma^{\hat 1\hat 7\hat 8}\right) + i c_{(3)}^{\mathrm{RR}}\left(\gamma^{\hat 5\hat 6\hat 9} - \gamma^{\hat 7\hat 8\hat 9}\right)\Big]\,\chi. \qquad (5.11)
\end{aligned}
$$

In the expression above χ is a 16-component spinor, the $\gamma^{\hat{\mu}}$ are 16×16 matrices defined in [57] and the coefficients $c_{(i)}$ are defined by

$$
\begin{aligned}
c_{(1)} &= \frac{\sigma^2-1}{A}\,, & c_{(2)} &= A\big(\sigma^2-1\big)^2,\\
c_{(\omega)} &= -\frac{1}{2\sqrt{A}}\,, & c^{\rm RR}_{(1)} &= -\frac{1}{4\sigma}\sqrt{A}\left(\sigma^2-1\right),\\
c^{\rm RR}_{(3)} &= -\frac{(2\sigma+A)\sqrt{\sigma^2-1}}{4\sigma\sqrt{A}}\,, & c^{\rm NSNS}_{(3)} &= \frac{\sqrt{A\left(\sigma^2-1\right)}}{4\sigma},\\
c^{\rm RR}_{(5)} &= \frac{4\sigma-\left(\sigma^2-1\right)A}{4\sigma\sqrt{A}}\,,
\end{aligned}
$$

where the $c_{(1)}$ and $c_{(2)}$ terms are the kinetic terms, the term with $c_{(\omega)}$ comes from the spin connection, $c^{\rm NSNS}_{(3)}$ is the contribution of the NSNS three-form, while the coefficients $\{c^{\rm RR}_{(1)}, c^{\rm RR}_{(3)}, c^{\rm RR}_{(5)}\}$ are the contributions of the RR fluxes.

Since we want the coefficient in front of ∂_τ in (5.11) to be σ independent, we perform the following rescaling

$$
\chi \to \frac{1}{c_{(1)}^{1/4}}\chi = \left(\frac{A}{\sigma^2-1}\right)^{1/4}\chi .
$$

This rescaling of the sixteen spinor components compensates the scaling performed to the eight bosonic transversal fluctuations, leaving the measure of the partition function unchanged.

After rescaling and choosing a convenient basis for the $\gamma^{\hat{\mu}}$'s, it is possible to rewrite (5.11) in a 2×2 block diagonal form

$$
\begin{aligned}
L_F^{(2)} = 2\sqrt{h}\Bigg[&\sum_{j=1}^{4}\begin{pmatrix}\bar{\psi}_{2j-1} & \bar{\psi}_{2j}\end{pmatrix}\tau_3\mathcal{D}_0\begin{pmatrix}\psi_{2j-1}\\ \psi_{2j}\end{pmatrix}\\
&+\sum_{j=5}^{6}\begin{pmatrix}\bar{\psi}_{2j-1} & \bar{\psi}_{2j}\end{pmatrix}\tau_3\mathcal{D}_+\begin{pmatrix}\psi_{2j-1}\\ \psi_{2j}\end{pmatrix}\\
&+\sum_{j=7}^{8}\begin{pmatrix}\bar{\psi}_{2j-1} & \bar{\psi}_{2j}\end{pmatrix}\tau_3\mathcal{D}_-\begin{pmatrix}\psi_{2j-1}\\ \psi_{2j}\end{pmatrix}\Bigg],
\end{aligned}
$$

where

$$
\mathcal{D}_0 = \begin{pmatrix}\partial_\tau & -L^\dagger\\ -L & \partial_\tau\end{pmatrix}, \qquad \mathcal{D}_\pm = \begin{pmatrix}\partial_\tau \pm 1 \pm \frac{A}{\sigma} & -\mathcal{L}^\dagger\\ -\mathcal{L} & \partial_\tau \mp 1\end{pmatrix}, \tag{5.12}
$$

with the definitions

$$
\begin{aligned}
L &= A\sqrt{\sigma^2-1}\,\partial_\sigma, & L^\dagger &= -A\sqrt{\sigma^2-1}\,\partial_\sigma + \frac{2}{\sqrt{\sigma^2-1}}\,,\\
\mathcal{L} &= A\sqrt{\sigma^2-1}\,\partial_\sigma - \frac{A\sqrt{\sigma^2-1}}{2\sigma}\,, & \mathcal{L}^\dagger &= -A\sqrt{\sigma^2-1}\,\partial_\sigma + \frac{4\sigma - A\left(\sigma^2-1\right)}{2\sigma\sqrt{\sigma^2-1}}.
\end{aligned} \tag{5.13}
$$

In terms of the determinants $\mathcal{D}_0$ and $\mathcal{D}_\pm$, the fermionic contribution to the semiclassical partition function is given by

$$\det \mathcal{K}_F = \det{}^4\mathcal{D}_0 \det{}^2\mathcal{D}_+ \det{}^2\mathcal{D}_- . \tag{5.14}$$

5.2.3 Putting everything together

Combining the bosonic and fermionic contributions to the semiclassical partition function

$$\frac{\det^{1/2}\mathcal{K}_F}{\det^{1/2}\mathcal{K}_B} = \frac{\det^2\mathcal{D}_0 \, \det \mathcal{D}_+ \, \det \mathcal{D}_-}{\det^{3/2}\mathcal{K}_{\mathbf{x}} \det^{1/2}\mathcal{K}_\phi \det \mathcal{K}_{\mathbf{y}}^+ \det \mathcal{K}_{\mathbf{y}}^-}.$$

This expression can be greatly simplified. The first simplification comes from realizing that time reversal relates the following operators

$$\mathcal{K}_{\mathbf{y}}^\pm |_{\tau\to-\tau} = \mathcal{K}_{\mathbf{y}}^\mp, \qquad\qquad \mathcal{D}_\pm |_{\tau\to-\tau} = -\tau_3 \mathcal{D}_\mp \tau_3 .$$

Since the determinants are time reversal invariant, we have that

$$\det \mathcal{K}_{\mathbf{y}}^+ = \det \mathcal{K}_{\mathbf{y}}^-, \qquad\qquad \det \mathcal{D}_+ = \det \mathcal{D}_- . \tag{5.15}$$

An additional simplification comes from noticing that the bosonic operators $\mathcal{K}_{\mathbf{x}}$ and $\mathcal{K}_\phi$ can be written in terms of the operators defined in (5.13)

$$\mathcal{K}_{\mathbf{x}} = -\partial_\tau^2 + L^\dagger L, \qquad \mathcal{K}_\phi = -\partial_\tau^2 + L L^\dagger . \tag{5.16}$$

Thus, these two operators are isospectral as explained in section 4.3. This property, combined with the choice of boundary conditions, implies $\det \mathcal{K}_{\mathbf{x}} = \det \mathcal{K}_\phi$.

Additionally, we have that squaring the operator $\mathcal{D}_0$

$$(\tau_3 \mathcal{D}_0)^2 = -\begin{pmatrix} \mathcal{K}_{\mathbf{x}} & 0 \\ 0 & \mathcal{K}_\phi \end{pmatrix}$$

implies

$$\det{}^2\mathcal{D}_0 = \det \mathcal{D}_0^2 = \det \mathcal{K}_{\mathbf{x}} \det \mathcal{K}_\phi . \tag{5.17}$$

Taking all these considerations into account, the semiclassical string partition function simplifies considerably

$$\frac{\det^{1/2}\mathcal{K}_F}{\det^{1/2}\mathcal{K}_B} = \frac{\det^2\mathcal{D}_-}{\det^2\mathcal{K}_{\mathbf{y}}^+}.$$

The operators $\mathcal{K}_{\mathbf{y}}^\pm$ can also be expressed through the operators in (5.13)

$$\mathcal{K}_{\mathbf{y}}^\pm = -\partial_\tau^2 + \mathcal{L}\mathcal{L}^\dagger + \frac{A}{\sigma} + 1 \pm \frac{A}{\sigma}\,\partial_\tau .$$

Using the following identity for block matrices

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left(AD - BD^{-1}CD\right) \overset{\text{if } [C,D]=0}{=} \det\left(AD - BC\right),$$

we have that

$$\det \mathcal{K}_{\mathbf{y}}^{\mp} = \det\begin{pmatrix} \partial_\tau \pm 1 \pm \frac{A}{\sigma} & -\mathcal{L} \\ -\mathcal{L}^\dagger & \partial_\tau \mp 1 \end{pmatrix}. \tag{5.18}$$

From equations (5.12) and (5.18), we can rewrite the result for the Wilson line as

$$\langle W(C)\rangle_{\text{line}} = \mathrm{e}^{-S_{\text{String}}(X_{\text{cl}})} \frac{\det^2\left(\partial_\tau - \mathcal{H}_F\right)}{\det^2\left(\partial_\tau - \mathcal{H}_B\right)}, \tag{5.19}$$

where

$$\mathcal{H}_B = \begin{pmatrix} 1 + \frac{A}{\sigma} & \mathcal{L} \\ \mathcal{L}^\dagger & -1 \end{pmatrix}, \qquad \mathcal{H}_F = \begin{pmatrix} -1 & \mathcal{L} \\ \mathcal{L}^\dagger & 1 + \frac{A}{\sigma} \end{pmatrix}, \tag{5.20}$$

and the term in the exponent is given in (5.6).

5.3 The spectral problems

After Fourier expanding around the worldsheet τ direction, we are left with a one-dimensional problem. The spectral problem we are interested in corresponds to

$$\mathcal{H}_{B,F}\ \chi = E\ \chi. \tag{5.21}$$

In principle, the most general solution to such equation corresponds to a superposition of two solutions χ_i with $i \in \{1,2\}$. In the neighbourhood of $\sigma \to 1$ the solutions in each superposition will behave as

$$\lim_{\sigma\to 1} \chi_i = \begin{pmatrix} (\sigma-1)^{\alpha_i} \\ (\sigma-1)^{\beta_i} \end{pmatrix}, \tag{5.22}$$

where α_i and β_i are constants. In order for χ_i to be an eigenvalue of $\mathcal{H}_{B,F}$, it must satisfy equation (5.21) in the region of σ close to 1. The later imposes strict constraints on the α_i's and β_i's of each solution χ_i in the superposition, fixing their values. Our boundary condition amounts to choosing the solution χ_i that has larger α_i and β_i as physically it will be better behaved at the origin, while the other solution is potentially problematic in the limit $\sigma \to 1$.

As discussed in section 4.4, we are only interested in the continuos part of the spectrum which is characterized by oscillating functions of plane-wave behaviour for $\sigma \to \infty$. Exponential functions can not satisfy the boundary condition at $\sigma = 1$ while simultaneously being normalizable. The density of states of the oscillating functions is parametrized by the derivative of the phaseshifts, which at large $\sigma = R$ are subject to the quantization condition

$$pR + \delta(p) = \pi n \qquad\qquad \Rightarrow \qquad\qquad \frac{dn}{dp} = \frac{\delta'(p)}{\pi} + \frac{R}{\pi}. \tag{5.23}$$

To evaluate each functional determinant we consider the ratio of the determinant of the operator we are interested in, over the determinant of its asymptotic operator $\mathcal{H}_\infty = \lim_{\sigma\to\infty}\mathcal{H}_{B,F}$. The later is our regularization prescription for all determinants in the partition function. Introduction of the bosonic and fermionic asymptotic operators does not alter the partition function as it amounts to multiplying by 1 in a convenient way since

$$(\mathcal{H}_\infty)^2 = \left(-\frac{4}{9}\partial_\sigma^2 + 1\right)\mathbb{1}.$$

The spectral problem of the asymptotic operators is given by

$$\mathcal{H}_\infty\ \chi_\infty = E_\infty(p)\ \chi_\infty,$$

where the components of χ_∞ are plane waves with wave number p and

$$E_\infty(p) = \pm\sqrt{\frac{4}{9}p^2+1}$$

is the corresponding eigenvalue parametrized in terms of p. The quantization condition for the asymptotic operator is

$$pR = \pi n \qquad\qquad \Rightarrow \qquad\qquad \frac{dn}{dp} = \frac{R}{\pi}.$$

Consequently, the effective density of states entering the ratio of determinants will be given by $\delta'(p)/\pi$, while we parametrize the eigenvalues of $\mathcal{K}_{B,F}$ by $E(p) = E_\infty(p)$.

With this regularization prescription, the (regularized) determinants will be given by

$$\begin{aligned}\ln\det\left(\partial_\tau - \mathcal{H}\right) =& \frac{\mathrm{L}}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\left[\frac{d\delta^+(p)}{dp}\ln\left(i\omega - |E(p)|\right)\right.\\ &\left. + \frac{d\delta^-(p)}{dp}\ln\left(i\omega + |E(p)|\right)\right]\frac{dp}{\pi}d\omega,\end{aligned}$$

where the sign of $\pm|E(p)|$ comes from the sign in (5.21) and $\delta^{\pm}(p)$ denotes the corresponding phaseshift. Integration by parts of this equation leads to

$$\begin{aligned}\ln\det\left(\partial_\tau - \mathcal{H}\right) &= \frac{\mathrm{L}}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\left[\frac{\delta^+(p)}{i\omega - |E(p)|} - \frac{\delta^-(p)}{i\omega + |E(p)|}\right]\frac{d|E(p)|}{dp}\frac{dp}{\pi}d\omega\\ &= -\frac{L}{2\pi}\int\limits_{0}^{\infty}\left[\delta^+(p) + \delta^-(p)\right]\frac{d|E(p)|}{dp}dp,\end{aligned}$$

where in the last step we integrated over ω picking up a half-residue at infinity. Combining the expression above with the classical contribution to the string

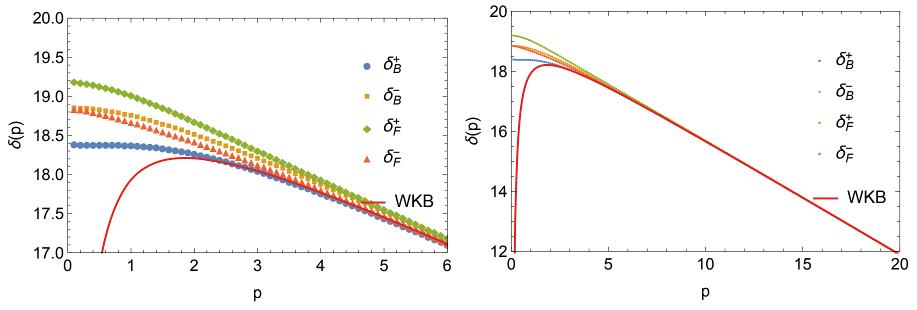

Figure 5.1. Numerical results for the phaseshifts.

partition function of (5.6) and replacing in (3.7), we find that

$$\ln \langle W(C) \rangle_{\text{line}} = \mathrm{L} \left(\frac{\sqrt{\lambda}}{2\pi} - \frac{1}{4} - \frac{\Delta}{4} \right), \tag{5.24}$$

where

$$\Delta = \frac{4\mathrm{L}}{\pi} \int_0^\infty \left[\delta_F^+ (p) + \delta_F^- (p) - \delta_B^+ (p) - \delta_B^- (p) \right] \frac{d\,|E\,(p)|}{dp} dp. \tag{5.25}$$

5.4 Numerics

Due to complicated expressions in the spectral problems, an analytical expression for the phaseshifts was not found. Instead, we resulted to calculate the $\delta^{\pm}_{B,F}(p)$ numerically and then numerically integrate (5.25).

The metric has a problematic behaviour at $\sigma = 1$, thus we take a small parameter $\epsilon = 10^{-6}$ and impose boundary conditions at $\sigma = 1 + \epsilon$ using our knowledge from (5.22)

$$\chi_i(\sigma = 1 + \epsilon) = \begin{pmatrix} \epsilon^{\alpha_i} \\ \epsilon^{\beta_i} \end{pmatrix}. \tag{5.26}$$

For a given value of p, we solve numerically for the eigenfunctions with the boundary condition (5.26) and at a large value of $\sigma = \sigma_{\max}$ we read the phaseshifts by fitting the eigenfunctions. We do this for $\sigma_{\max} = 10^3$ and for p in the range $[p_{\min} = 10^{-1}, p_{\max} = 50]$ in intervals of $\delta p = 10^{-1}$, resulting in approximately 500 points per phaseshift. The resulting curves for each phaseshift in terms of p are presented in figure 5.1.

After computing each phaseshift in terms of p, we numerically integrate obtaining

$$\Delta = 1.01 \pm 0.03. \tag{5.27}$$

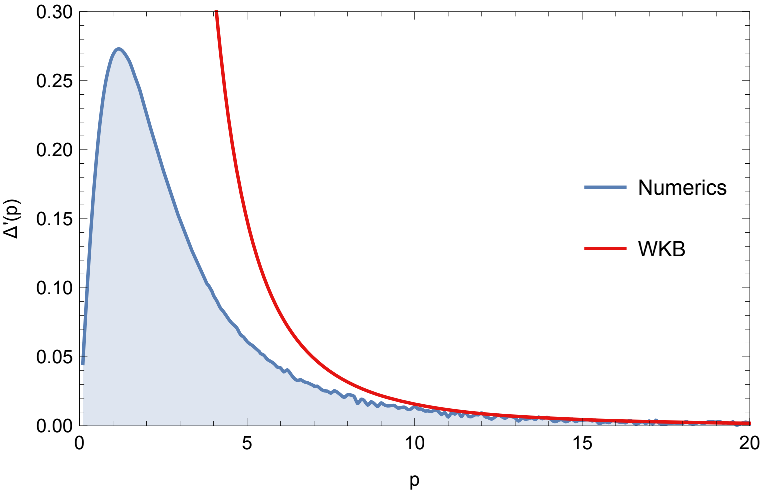

Figure 5.2. Integrand Δ' as a function of p. The area under the curve is Δ of equation (5.25).

This result combined with (5.24) matches the localization prediction (5.1).

The integrand in (5.25) tends to zero as p becomes larger as can be seen in figure 5.2, which makes the area Δ a finite quantity. To understand the behaviour of the integral at large p, in paper I we performed a WKB expansion of the phaseshifts

$$\delta(p) = p\delta_0 + \delta_1 + \frac{1}{p}\delta_2 + ... \qquad (p \to \infty)$$

and analytically evaluated the first four coefficients δ_i, showing explicit cancelation up to $\mathcal{O}(p^{-3})$. This non-trivial cancellation guarantees that Δ is well behaved at ∞ and it also explains the similar linear behaviour obtained for all $\delta^{\pm}_{B,F}(p)$ at large p.

6. Ratio of latitude and circular Wilson loops in $AdS_5 \times S^5$

The ratio between the expectation value of a latitude Wilson loop and a circular Wilson loop in $\mathcal{N} = 4$ SYM at strong coupling is according to localization

$$\ln \frac{\langle W(\theta_0)\rangle}{\langle W(0)\rangle} = \sqrt{\lambda}\,(\cos\theta_0 - 1) - \frac{3}{2}\ln\cos\theta_0 + \mathcal{O}\left(\lambda^{-1/2}\right), \tag{6.1}$$

where $\theta_0 \in [0, \frac{\pi}{2})$ denotes the polar angle in a $S^2 \subset S^5$ and λ is the 't Hooft coupling constant.

The string theory calculation of (6.1) in $AdS_5 \times S^5$ is the main result of paper II. Here we present how to do this computation following the formalism presented in section 3.3 and using the functional determinant techniques of sections 4.2 and 4.4.

As discussed in section 3.4.3, the string configuration dual to the latitude Wilson loop is described by a string solution ending in a circle at the boundary of AdS and extending into a latitude in the five-sphere.

Following the gauge fixing prescription discussed in section 3.3, the world-sheet metric is chosen to be the metric induced by the classical solution (3.13)

$$ds^2_{\text{w.s.}} = \Omega^2\left(d\tau^2 + d\sigma^2\right), \tag{6.2}$$

where $\tau \in [0, 2\pi]$, $\{\sigma, \sigma_0\} \in [0, \infty)$ and

$$\Omega^2 = \frac{1}{\sinh^2\sigma} + \frac{1}{\cosh^2(\sigma + \sigma_0)}.$$

For convenience we use the parameter σ_0 instead of θ_0, where the two are related by

$$\cos\theta_0 = \tanh\sigma_0. \tag{6.3}$$

In terms of σ_0, the localization prediction for the ratio is given by

$$\ln \frac{\langle W(\sigma_0)\rangle}{\langle W(\infty)\rangle} = \sqrt{\lambda}\,(\tanh\sigma_0 - 1) - \frac{3}{2}\ln(\tanh\sigma_0) + \mathcal{O}\left(\lambda^{-1/2}\right). \tag{6.4}$$

In section 6.1 we present the contribution coming from the classical string partition function. Section 6.2 concerns the contributions to the semiclassical partition function and their calculation. The different pieces are collected in section 6.3. Several technical details are omitted in the present text, the reader interested in them is suggested to see paper II.

6.1 The classical partition function

The classical contribution to the string partition function is calculated using the regularization prescription of section 3.3.1

$$S_{\mathrm{B}}\left(X_{\mathrm{cl}}(\sigma_0)\right) = \frac{\sqrt{\lambda}}{2\pi}\int\limits_{\mathrm{reg}} \Omega^2 d\tau d\sigma = -\sqrt{\lambda}\tanh\sigma_0, \tag{6.5}$$

where the integration in τ is over the interval $[0, 2\pi]$ while for σ the interval is $[\epsilon, \infty)$. As discussed in section 3.3.1, the $1/\epsilon$ divergences are dropped from the final result.

For the ratio of the latitude and circular Wilson loops, the relevant quantity is given by

$$S_{\mathrm{B}}\left(X_{\mathrm{cl}}\left(\sigma_0\right)\right) - S_{\mathrm{B}}\left(X_{\mathrm{cl}}\left(\infty\right)\right) = -\sqrt{\lambda}\left(\tanh\sigma_0 - 1\right), \tag{6.6}$$

which perfectly matches the localization prediction (6.4) at order $\sqrt{\lambda}$.

6.2 The semiclassical partition function

As discussed in section 3.3.2, contributions to the semiclassical partition function are the result of considering second order fluctuations around the classical string solution. After Gaussian integration, the semiclassical piece is written in terms of the determinants of differential operators [12, 13, 14]

$$Z_{\text{1-loop}}(\sigma_0) = \frac{\det^{1/2}\mathcal{K}_B\left(\sigma_0\right)}{\det^{1/2}\mathcal{K}_F\left(\sigma_0\right)} = \frac{\det^2\mathcal{D}_+ \det^2\mathcal{D}_-}{\det^{3/2}\mathcal{K}_1 \det^{3/2}\mathcal{K}_2 \det^{1/2}\mathcal{K}_{3+} \det^{1/2}\mathcal{K}_{3-}}, \tag{6.7}$$

where the untilded operators on the right are defined through

$$\mathcal{K} = \frac{1}{\Omega^2}\widetilde{\mathcal{K}}, \qquad \mathcal{D} = \frac{1}{\Omega^{\frac{3}{2}}}\widetilde{\mathcal{D}}\Omega^{\frac{1}{2}}, \tag{6.8}$$

and the tilded operators

$$\widetilde{\mathcal{K}}_1 = -\partial_\tau^2 - \partial_\sigma^2 + \frac{2}{\sinh^2\sigma}, \tag{6.9}$$

$$\widetilde{\mathcal{K}}_2 = -\partial_\tau^2 - \partial_\sigma^2 - \frac{2}{\cosh^2\left(\sigma+\sigma_0\right)}, \tag{6.10}$$

$$\begin{aligned}\widetilde{\mathcal{K}}_{3\pm} = {} & -\partial_\tau^2 - \partial_\sigma^2 \pm 2i\left(\tanh\left(2\sigma+\sigma_0\right) - 1\right)\partial_\tau \\ & + \left(\tanh\left(2\sigma+\sigma_0\right) - 1\right)\left(1 + 3\tanh\left(2\sigma+\sigma_0\right)\right),\end{aligned} \tag{6.11}$$

$$\begin{aligned}\widetilde{\mathcal{D}}_\pm = {} & i\partial_\sigma\tau_1 - \left[i\partial_\tau \mp \frac{1}{2}\left(1 - \tanh\left(2\sigma+\sigma_0\right)\right)\right]\tau_2 \\ & + \frac{1}{\Omega\sinh^2\sigma}\tau_3 \mp \frac{1}{\Omega\cosh^2\left(\sigma+\sigma_0\right)}.\end{aligned} \tag{6.12}$$

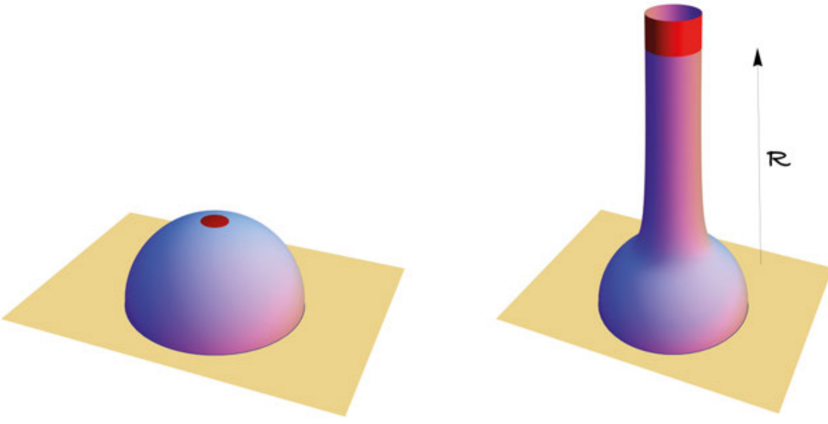

Figure 6.1. Graphical representation of the minimal surface of the worldsheet in the induced metric (left) and on the flat metric of the cylinder (right). The region $\sigma > R$ of area s is removed by the regularization scheme.

The operator $\mathcal{K}_1$ physically corresponds to three fluctuation modes in AdS, $\mathcal{K}_2$ to three fluctuation modes along the five-sphere, while the $\mathcal{K}_{3\pm}$ result from the mixing of one fluctuation mode from the sphere and one from AdS. The operator $\mathcal{D}_\pm$ is the operator resulting from the fermionic action after κ symmetry gauge fixing. In the present problem, the bosonic determinants have periodic boundary conditions in the τ coordinate, while the fermionic operators have anti-periodic boundary conditions [9, 12, 13].

Despite the untilded operators being the original operators appearing when expanding the Green-Schwarz action, for the purposes of computation it is easier to evaluate the determinants of the tilded operators. The tilded and untilded operators are connected through the formalism explained in section 4.2 and have the inner products

$$\langle \phi_1 \,|\, \phi_2 \rangle = \int d\tau\, d\sigma\, \Omega^2 \phi_1^\dagger \phi_2, \qquad \widetilde{\langle \phi_1 \,|\, \phi_2 \rangle} = \int d\tau\, d\sigma\, \phi_1^\dagger \phi_2, \tag{6.13}$$

respectively. The transformation from the tilded to the untilded operators can be seen as a conformal transformation. This transformation takes the string from having the metric (6.2) of the disk, to the flat metric of an infinite cylinder. This seemingly innocent transformation changes the topology of the worldsheet since the point $\sigma = \infty$ is regular in the induced metric but not in the flat one as seen in figure 6.1. Going from the spectral problem of the disk to the cylinder requires an IR regularization, and it is an anomaly from this regularization what solves the reported discrepancies with the localization result [12, 13].

The procedure for evaluating the determinant of each untilded operator is done in several steps

$$\det \mathcal{K} = \left(\frac{\det \mathcal{K}}{\det \widetilde{\mathcal{K}}}\right)_{\text{anom.}} \left(\frac{\det \widetilde{\mathcal{K}}}{\det \widetilde{\mathcal{K}}_\infty}\right)_{\text{cyl.}} \det \widetilde{\mathcal{K}}_\infty . \tag{6.14}$$

First, we calculate the ratio of the untilded and tilded determinants by employing the results presented in section 4.2, this corresponds to the first factor in the r.h.s. of equation (6.14). Then, we proceed to evaluate the determinant of the tilded operator by considering its ratio with the determinant of its asymptotic operator. The later corresponds to the second factor in the r.h.s. of equation (6.14) and computationally amounts to evaluating the relative phaseshifts between the eigenfunctions of the two operators. Naturally, introducing the asymptotic operator as regulator introduces the third factor in the r.h.s. of (6.14). The asymptotic operators of (6.9), (6.10), (6.11) and (6.12) are given by

$$\widetilde{\mathcal{K}}_\infty = -\partial_\tau^2 - \partial_\sigma^2, \qquad \widetilde{\mathcal{D}}_\infty = i\tau_1 \partial_\sigma - i\tau_2 \partial_\tau , \tag{6.15}$$

and satisfy $\left(\widetilde{\mathcal{D}}_\infty\right)^2 = \mathbb{1}\widetilde{\mathcal{K}}_\infty$.

However, due to fermions and bosons having symmetric and antisymmetric boundary conditions, the determinants of bosonic and fermionic asymptotic operators do not cancel, leaving the semiclassical string partition function $Z(\sigma_0)$ explicitly depending on the IR regulator. It is the remnant introduced by this regulator what solves the previously found mismatch when considering the ratio of the latitude and circle partition functions.

As discussed previously, the semiclassical partition function of the string dual to the latitude Wilson loop has contributions of three types

$$Z_{\text{1-loop}}(\sigma_0) = Z_{\text{CF}}(\sigma_0)\, Z_\delta(\sigma_0)\, Z_\infty(\sigma_0) , \tag{6.16}$$

the contribution from the ratio of tilded and untilded operators $Z_{\text{CF}}(\sigma_0)$, the evaluation of the functional determinants in terms of phaseshifts $Z_\delta(\sigma_0)$ and the regulators coming from the asymptotic operators $Z_\infty(\sigma_0)$. We now proceed to evaluate each of these contributions.

6.2.1 The phaseshifts

We calculate the determinant of the tilded operators by using their corresponding asymptotic operators as regulators. This contribution is given by

$$Z_\delta(\sigma_0) = \prod_{\beta=1,2} \left(\frac{\det \widetilde{\mathcal{K}}_\infty}{\det \widetilde{\mathcal{K}}_\beta}\right)^{3/2} \prod_{\alpha=+,-} \left(\frac{\det \widetilde{\mathcal{D}}_\alpha}{\det \widetilde{\mathcal{D}}_\infty}\right)^2 \left(\frac{\det \widetilde{\mathcal{K}}_\infty}{\det \widetilde{\mathcal{K}}_{3\alpha}}\right)^{1/2} \tag{6.17}$$

and is calculated through the phaseshift method described in section 4.4.

The spectral problem we are interested in corresponds to

$$\widetilde{\mathcal{K}}\psi = E\psi. \tag{6.18}$$

As was explained in section 4.4, we are interested in the continuum spectrum, where the eigenfunctions are oscillating functions since these will be normalizable and satisfy the boundary conditions at the origin. We impose for the eigenfunctions $\psi(\sigma = 0) = 0$ and at large $\sigma = R$ the quantization condition

$$pR + \delta(p,\omega) = \pi n \qquad \Rightarrow \qquad \frac{dn}{dp} = \frac{1}{\pi}\partial_p \delta(p,\omega) + \frac{R}{\pi},$$

where p is the wave number of ψ, ω is the frequency resulting from Fourier expanding in τ, n is an integer and $\delta(p,\omega)$ is the phaseshift which in general can depend on ω. The dependence of the phaseshift on ω is necessary for our calculations as the operators $\widetilde{\mathcal{K}}_{3\pm}$ have a linear derivative in τ. This linear derivative can be seen as an ω dependent potential term after Fourier expanding.

The spectral problem for the asymptotic operator is given by

$$\widetilde{\mathcal{K}}_\infty \; \psi_\infty = (-\partial_\tau^\sigma - \partial_\sigma^2)\psi_\infty = E_\infty(p,\omega)\; \psi_\infty,$$

where p denotes the wave number of the plane wave ψ_∞. It is easy to see that Fourier expansion implies

$$E_\infty(p,\omega) = \omega^2 + p^2,$$

while the quantization condition for the asymptotic operators is

$$pR = \pi n \qquad \Rightarrow \qquad \frac{dn}{dp} = \frac{R}{\pi}. \tag{6.19}$$

Parametrizing the eigenvalues $E = E_\infty(p,\omega)$, the prescription from section 4.4 for determinants results in

$$\begin{aligned}\ln\frac{\det\widetilde{\mathcal{K}}}{\det\widetilde{\mathcal{K}}_\infty} &= \sum_\omega \int_0^\infty \ln\left(\omega^2 + p^2\right)\partial_p\delta\left(p,\omega\right)\frac{dp}{\pi} \\ &= -2\sum_\omega \int_0^\infty \frac{p}{\omega^2+p^2}\delta\left(p,\omega\right)\frac{dp}{\pi}, \end{aligned}\tag{6.20}$$

where $\omega \in \mathbb{Z}$ for bosons and $\omega \in \mathbb{Z} + 1/2$ for fermions. To carry out the summation over ω it is convenient to use Matsubara frequencies, which amounts to the replacements

$$\sum_{\omega\in\mathbb{Z}} 1 \Rightarrow \pi \oint_C \frac{d\omega}{2\pi i}\cot\pi p, \qquad \sum_{\omega\in\mathbb{Z}+1/2} 1 \Rightarrow -\pi \oint_C \frac{d\omega}{2\pi i}\tan\pi p,$$

where the contour C is presented in figure 6.2 and encloses the real axis in the complex plane. After modifying the contour to encompass the upper and

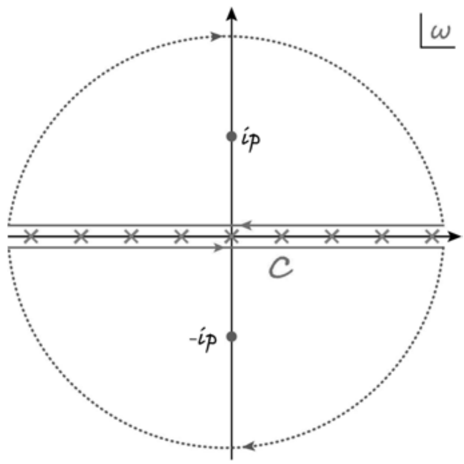

Figure 6.2. Contours of integration used for summing Matsubara frequencies.

lower arcs, complex integration over ω picks up poles at $\omega = \pm ip$. Denoting $\delta_{\pm}(p) = \delta(p, \pm ip)$, the resulting determinants for periodic and anti-periodic boundary conditions are

$$\ln \frac{\det \tilde{\mathcal{K}}}{\det \widetilde{\mathcal{K}}_{\infty}} = -\int_0^{\infty} \coth \pi p \;\; [\delta_+ (p) + \delta_- (p)] dp \qquad \text{(periodic)}, \tag{6.21}$$

$$\ln \frac{\det \tilde{\mathcal{K}}}{\det \widetilde{\mathcal{K}}_{\infty}} = -\int_0^{\infty} \tanh \pi p \;\; [\delta_+ (p) + \delta_- (p)] dp \qquad \text{(anti-periodic)}, \tag{6.22}$$

respectively. We will use these expression to calculate the determinants entering equation (6.17).

For explicit computation of the phaseshifts, the spectral problem (6.18) reduces to

$$\widetilde{\mathcal{K}}|_{\omega=\pm ip} \; \psi = 0.$$

The resulting eigenfunctions for each operator are presented in paper II. The corresponding phaseshifts with their multiplicities give

$$\ln \frac{Z_\delta (\sigma_0)}{Z_\delta (\infty)} = \int_0^{\infty} dp \left[-4 \left(\arctan \frac{p}{\frac{1}{2} + \tanh \sigma_0} - \arctan \frac{2p}{3} \right) \tan \pi p \right.$$
$$\left. + \left(\arctan \frac{p}{1 + \tanh \sigma_0} - \arctan \frac{p}{2} + 3 \arctan \frac{p}{\tanh \sigma_0} - 3 \arctan p \right) \coth \pi p \right].$$

To perform this integral we will first differentiate with respect to σ_0 on both sides, integrate over p, and then integrate over σ_0. The final integration constant can be fixed by a simple argument, as we will see later. Differentiating the expression above we obtain

$$\begin{aligned}\frac{d}{d\sigma_0}\ln\frac{Z_\delta(\sigma_0)}{Z_\delta(\infty)} &= \int\limits_0^\infty \frac{dp}{\cosh^2\sigma_0}\left[\frac{4\tanh(\pi p)\,p}{p^2+\left(\frac{1}{2}+\tanh\sigma_0\right)^2}\right.\\ &\qquad\left.-\frac{\coth(\pi p)\,p}{p^2+(1+\tanh\sigma_0)^2}-\frac{3\coth(\pi p)\,p}{p^2+\tanh^2\sigma_0}\right]\\ &= \frac{1}{2}-\frac{3}{2}\coth\sigma_0+\tanh\sigma_0.\end{aligned}$$

Integrating in σ_0 we obtain

$$\ln\frac{Z_\delta(\sigma_0)}{Z_\delta(\infty)} = -\frac{3}{2}\ln(\tanh\sigma_0)+\ln\left(\sqrt{1+\tanh\sigma_0}\right)+\mathrm{C},$$

where C is a constant of integration. The later can be easily fixed by requiring

$$\lim_{\sigma_0\to\infty}\left[\ln\frac{Z_\delta(\sigma_0)}{Z_\delta(\infty)}\right]=0,$$

obtaining the final result

$$\ln\frac{Z_\delta(\sigma_0)}{Z_\delta(\infty)} = -\frac{3}{2}\ln(\tanh\sigma_0)+\ln\sqrt{\frac{1+\tanh\sigma_0}{2}}. \tag{6.23}$$

The result above differs from the localization prediction (6.4) by an extra term and matches the results of the Gel'fand-Yaglom computations [12, 13].

6.2.2 Invariant regulators

The contribution resulting from using the asymptotic tilded operators as regulators amounts to

$$Z_\infty(\sigma_0)=\frac{\det^4\widetilde{\mathcal{D}}_\infty}{\det^4\widetilde{\mathcal{K}}_\infty}=\frac{\prod\limits_{\omega\in\mathbb{Z}+1/2}\det^4\left(\omega^2-\partial_\sigma^2\right)}{\prod\limits_{\omega\in\mathbb{Z}}\det^4\left(\omega^2-\partial_\sigma^2\right)}.$$

Using the phaseshift formalism and the asymptotic density of states (6.19), the above can be written as

$$\begin{aligned}\ln Z_\infty(\sigma_0) &= \frac{4R}{\pi}\left[\sum_{\omega\in\mathbb{Z}+1/2}\int\limits_0^\infty\ln\left(\omega^2+p^2\right)dp-\sum_{\omega\in\mathbb{Z}}\int\limits_0^\infty\ln\left(\omega^2+p^2\right)dp\right]\\ &= -8R\int\limits_0^\infty p\,(\tanh\pi p-\coth\pi p)\,dp = R,\end{aligned} \tag{6.24}$$

where we used Matsubara frequencies and contour integration.[1]

Now that we understand how does the partition function depend on the regulator R, let us examine the later in more detail. The regulator R appears by imposing Dirichlet boundary conditions on the eigenfunctions at a large but finite value of $\sigma = R$. Geometrically, as seen in figure 6.1, employing this regulator amounts to removing part of the minimal surface of the string. Naturally, the area of the worldsheet removed depends on σ_0 as so does the worldsheet metric. Thus, taking a generic σ_0-independent regulator has no invariant meaning by itself as we need a diffeomorphism-invariant regularization.

The area s removed by this regularization procedure is given by

$$s = \int_0^{2\pi} \int_R^{\infty} \Omega^2 d\sigma d\tau \simeq 4\pi \left(1 + e^{-2\sigma_0}\right) e^{-2R}.$$

Choosing this area to remain invariant in the regularization procedure amounts to having a σ_0-dependent regulator

$$R(\sigma_0) = -\frac{1}{2} \ln\left(1 + \tanh \sigma_0\right) - \frac{1}{2} \ln\left(\frac{s}{8\pi}\right).$$

Consequently, the net contribution of IR regulators to the ratio of latitude and circle partition functions is given by

$$\ln \frac{Z_\infty(\sigma_0)}{Z_\infty(\infty)} = R(\sigma_0) - R(\infty) = -\ln \sqrt{\frac{1 + \tanh \sigma_0}{2}}, \tag{6.25}$$

which cancels the second term in (6.23).

6.2.3 Conformal factors

The term accounting for the ratio of tilded and untilded operators is explicitly given by

$$Z_{\mathrm{CF}}(\sigma_0) = \prod_{\beta=1,2} \left(\frac{\det \widetilde{\mathcal{K}}_\beta}{\det \mathcal{K}_\beta}\right)^{3/2} \prod_{\alpha=+,-} \left(\frac{\det \mathcal{D}_\alpha}{\det \widetilde{\mathcal{D}}_\alpha}\right)^2 \left(\frac{\det \widetilde{\mathcal{K}}_{3\alpha}}{\det \mathcal{K}_{3\alpha}}\right)^{1/2}.$$

To calculate this quantity we use equations (4.17) and (4.21) with the identification that $\alpha = 1$ corresponds to the untilded operators, while $\alpha = 0$ the tilded ones. Additional identifications needed are

$$\phi = -\ln \Omega, \qquad a_\pm = \frac{1}{\Omega \sinh^2 \sigma}, \qquad v_\pm = \mp \frac{1}{\Omega \cosh^2(\sigma + \sigma_0)},$$
$$E_1 = \frac{2}{\sinh^2 \sigma}, \qquad E_2 = -\frac{2}{\cosh^2(\sigma + \sigma_0)}, \qquad E_{3\pm} = -\frac{2}{\cosh^2(2\sigma + \sigma_0)}.$$

[1] Alternatively, a short computation of these integrals consists of comparing the integrals in the first lines of (6.20) and (6.24). After the identification $R = \delta'(p) \to \delta(p) = Rp$, we can then use the results (6.21) and (6.22) for periodic and anti-periodic frequencies.

It is easy to see that the surface terms in (4.17) and (4.21) which contribute to $Z_{\rm CF}\left(\sigma_0\right)$ cancel between bosons and fermions. For the bulk terms we have that the integrand is proportional to

$$\left[\frac{2}{3}\phi\partial_\sigma^2\phi - \phi\left(\frac{3}{2}E_1 + \frac{3}{2}E_2 + E_{3\pm}\right)\right] + 4\left[\frac{1}{12}\phi\partial_\sigma^2\phi + \phi\left(a_\pm^2 - v_\pm^2\right)\right] = 0.$$

Consequently, we obtain

$$\ln Z_{\rm CF}(\theta_0) = 0. \tag{6.26}$$

This result is to be expected as conformal anomaly cancelations are important in string theory, making this a consistency check of our calculation.

6.3 Putting everything together

Collecting the classical contributions (6.6) and the semiclassical contributions (6.23), (6.25), (6.26), we obtain the following result for the ratio of latitude and circular Wilson loops

$$\ln\frac{\langle W\left(\sigma_0\right)\rangle}{\langle W\left(\infty\right)\rangle} = \sqrt{\lambda}\left(\tanh\sigma_0 - 1\right) - \frac{3}{2}\ln\left(\tanh\sigma_0\right) + O\left(\lambda^{-1/2}\right), \tag{6.27}$$

obtaining perfect agreement with the localization prediction (6.4).

7. Conclusions and open problems

In the present thesis we present recent developments in the computation of Wilson loops in string theory. Wilson loops play an important role in gauge theories and particularly in the gauge-string duality as a physical description exists at both sides. The later makes these observables an ideal probe for the duality as they can in principle be computed both in field theory and string theory. In field theory, localization has lead to the computation of several Wilson loop configurations at all orders in the coupling. In the string theory side the picture is much less clear as only leading order contributions at strong coupling are fully understood. The present thesis focuses on next to leading order contributions by computing the semiclassical string partition function of two Wilson loop configurations.

In paper I we computed the semiclassical string partition function for a straight line in the Pilch-Warner background, obtaining perfect agreement with localization predictions for $\mathcal{N} = 2^*$. This 1-loop computation is first of its kind for nonconformal theories and provides a quantitative test of $\mathcal{N} = 2^*$ holography. Additionally, this calculation shades light on the role played by the Fradkin-Tseytlin term in the Green-Schwarz action as its contribution was necessary to reproduce the field theory result. Due to the complicated expressions involved, it was necessary to resort to numerics for the final answer. However, the simplicity of the final answer suggests that an analytical calculation may be possible. The techniques developed for this calculation can also be applied to other Wilson loop configurations, as done in paper I, where the straight line result for $AdS_5 \times S^5$ was obtained analytically in a simple and elegant manner.

Paper II concerns the holographic calculation of the ratio of latitude and circular Wilson loops in $AdS_5 \times S^5$ at strong coupling. An anomaly related to the conformal transformation from the disk to the cylinder was shown to be responsible for the previously found discrepancies with the field theory result at 1-loop [12, 13]. The successful matching with the localization prediction in paper II suggests that the assumptions of ghost/longitudinal mode cancelations between the two loops are correct, as well as there being no conformal anomaly contributions or additional parameter dependent measure contributions.

Despite progress in the understanding of these calculations, many problems remain unsolved. Wilson loops are well defined finite quantities in field theory, thus, calculation of individual Wilson loops in string theory should be well defined. A long standing problem along these lines is the string theory 1-loop computation of the 1/2 BPS circular Wilson loop. A first piece of the puzzle would consist of a proper understanding of the boundary conditions for the spectral problems at hand, in particular for longitudinal modes and ghosts. Explicit calculation shows that the second order elliptic operators involved can be written in terms of integrable potentials like the Pöschl-Teller or the Rosen-Morse potential. Thus, using techniques from integrable systems it might be possible to study the spectrum of these differential operators and to track large parts of the calculation. A full solution to the problem would additionally require to "tame" the divergences present in the calculation, as the physical problem in question must have a finite solution.

For 1-loop string Wilson loop computations in general, the role played by zero-modes in the ghost determinant must also be carefully considered. This issue has been largely ignored in the present thesis and in previous computations but must be eventually addressed. A possible approach to the problem comes from the use of collective coordinates. More mathematically powerful tools could be explored, in particular index theorems which allow to connect the spectral properties of differential operators with topology.

Finally, an even more complicated question is the development of techniques that would allow for the calculation of Wilson loops beyond 1-loop order in string theory. Despite the difficult challenges that this implies, exact results at all orders from localization and the gauge-string duality suggest that in string theory a similar mathematical mechanism may exist, of which our current perturbative techniques are the tip of the iceberg.

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Svensk sammanfattning

Gauge-strängteori-korrespondensen har varit en av de mest spännande områdena inom teoretisk fysik och hävdar att vissa kvantfältteorier har en strängteoridual. Den överraskande naturen hos denna dualitet är sådan att ena sidan av korrespondensen är starkt växelverkande medan den andra teorin är svagt växelverkande. På så sätt kan vi få nya insikter om starkt kopplade fältteorier genom att studera svagt kopplad strängteori. Detta har stor betydelse då kvantfältteorier är grunden för vår nuvarande förståelse av universum samtidigt som att många av deras egenskaper vid stark växelverkan inte är kända, då störningsräkning inte längre är tillämpbart i detta parameterområde.

Wilsonloopar är observabler som dyker upp i många områden inom teoretisk fysik och spelar en viktig roll i studiet av starkt växelverkande fältteorier, i egenskap av en orderparameter för confinement. Inom gauge-sträng-korrepsondensens ramverk spelar dessa observabler en betydande roll då de har en känd fysikalisk beskrivning på båda sidor om dualiteten. Detta gör Wilsonloopar till ett unikt testverktyg för dualiteten eftersom att samma storhet kan räknas ut på båda sidor. Därutöver kan vidgad förståelse av strängteoriräkningarna ge oss nya insikter inom den duala fältteorin vid stark växelverkan.

Då strängteorin är svagt växelverkande kan vi i princip använda störningsräkning som metod i beräkningarna av Wilsonloopar. Vår nuvarande förståelse av strängteori i krökta rumtider och våra beräkningstekniker har emellertid bara varit framgångsrika till ledande ordningen och, i några speciella fall, andra ordningen i störningsräkning. Till andra ordningen i störningsräkningen reduceras beräkningen av Wilsonloopar till ett antal determinanter av andra ordningens differentialoperatorer. Många problem har träffats på i den här typen av uträkningar: motstridiga resultat jämfört med fältteorin, divergenser, oklara randvillkor, etc. Forskningen som presenteras i denna avhandling syftar till en bättre förståelse av dessa problem och att utveckla tekniker för strängteoriräkningen av dessa storheter bortom ledande ordningen.

Tack vare uppseendeväckande framsteg från supersymmetrisk lokalisering inom fältteori så finns det exakta förutsägelser till alla ordningar för vissa Wilsonloopkonfigurationer. Dessa konfigurationer utgör en perfekt testbana för att utveckla nya beräkningstekniker för Wilsonloopar i strängteori. Förhoppningen är att lärdomarna dragna från dessa beräkningar kommer att leda till nya förutsägelser om starkt växelverkande fältteorier.

Den första beräkningen som presenteras i denna avhandling behandlar Wilsonlinjen i $\mathcal{N} = 2^*$. Denna teori är en massiv deformation av $\mathcal{N} = 4$ SYM och har därför ingen konform symmetri. Den strängteoretiska dualen till $\mathcal{N} = 2^*$ är typ IIB Pilch-Warner-bakgrunden. I artikel I beräknade vi 1-loopsbidraget till den räta Wilsonlinjen genom nya tekniker för beräkningen av funktionaldeterminanter och fann perfekt överensstämmelse med lokaliseringsförutsägelsen vilket visade dess divergensfria natur. Detta framgångsrika 1-looptest av gauge-sträng-dualiteten var det första av sitt slag för icke-konforma teorier och öppnar dörren för beräkningar i mer realistiska teorier.

Den andra beräkningen som presenteras här är kvoten mellan en latitud-Wilsonloop och en cirkulär Wilsonloop i $\mathcal{N} = 4$ SYM. Strängteoriberäkningen av denna storhet i $AdS_5 \times S^5$ har tidigare inte lyckats stämma med lokaliseringsresultat. I artikel II löste en noggrann studie av IR-anomalin relaterad till divergensen i den konforma faktorn detta problem.

Trots framstegen i förståelsen av Wilsonloopar i strängteori återstår många öppna frågor och problem. Bland dem finns 1-loopberäkningar av den cirkulära och en k-lindad Wilsonloop i $AdS_5 \times S^5$. Andra olösta tekniska frågor relaterar till spökpartiklars/longitudinella moders inverkan, likväl som siktet på högre ordningar i störningsräkningar. Exakta resultat från lokalisering tillsammans med gauge-sträng-dualiteten pekar på att våra nuvarande tekniker kanske bara är tippen av isberget.

References

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* **38** (1999) 1113–1133, hep-th/9711200. [Adv. Theor. Math. Phys.2,231(1998)].

[2] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," *Commun. Math. Phys.* **313** (2012) 71–129, 0712.2824.

[3] N. Drukker, D. J. Gross, and A. A. Tseytlin, "Green-Schwarz string in AdS(5) x S**5: Semiclassical partition function," *JHEP* **04** (2000) 021, hep-th/0001204.

[4] K. Pilch and N. P. Warner, "N=2 supersymmetric RG flows and the IIB dilaton," *Nucl. Phys.* **B594** (2001) 209–228, hep-th/0004063.

[5] K. Pilch and N. P. Warner, "Generalizing the N=2 supersymmetric RG flow solution of IIB supergravity," *Nucl. Phys.* **B675** (2003) 99–121, hep-th/0306098.

[6] X. Chen-Lin, J. Gordon, and K. Zarembo, "$\mathcal{N} = 2*$ super-Yang-Mills theory at strong coupling," *JHEP* **11** (2014) 057, 1408.6040.

[7] E. I. Buchbinder and A. A. Tseytlin, "1/N correction in the D3-brane description of a circular Wilson loop at strong coupling," *Phys. Rev.* **D89** (2014), no. 12 126008, 1404.4952.

[8] J. K. Erickson, G. W. Semenoff, and K. Zarembo, "Wilson loops in N=4 supersymmetric Yang-Mills theory," *Nucl. Phys.* **B582** (2000) 155–175, hep-th/0003055.

[9] M. Kruczenski and A. Tirziu, "Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling," *JHEP* **05** (2008) 064, 0803.0315.

[10] C. Kristjansen and Y. Makeenko, "More about One-Loop Effective Action of Open Superstring in $AdS_5 \times S^5$," *JHEP* **09** (2012) 053, 1206.5660.

[11] R. Bergamin and A. A. Tseytlin, "Heat kernels on cone of AdS_2 and k-wound circular Wilson loop in $AdS_5 \times S^5$ superstring," *J. Phys.* **A49** (2016), no. 14 14LT01, 1510.06894.

[12] V. Forini, V. Giangreco M. Puletti, L. Griguolo, D. Seminara, and E. Vescovi, "Precision calculation of 1/4-BPS Wilson loops in $\mathrm{AdS}_5 \times S^5$," *JHEP* **02** (2016) 105, 1512.00841.

[13] A. Faraggi, L. A. Pando Zayas, G. A. Silva, and D. Trancanelli, "Toward precision holography with supersymmetric Wilson loops," *JHEP* **04** (2016) 053, 1601.04708.

[14] V. Forini, A. A. Tseytlin, and E. Vescovi, "Perturbative computation of string one-loop corrections to Wilson loop minimal surfaces in $\mathrm{AdS}_5 \times \mathrm{S}^5$," *JHEP* **03** (2017) 003, 1702.02164.

[15] J. Aguilera-Damia, A. Faraggi, L. A. Pando Zayas, V. Rathee, and G. A. Silva, "Zeta-function Regularization of Holographic Wilson Loops," `1802.03016`.
[16] M. Cvetic, H. Lu, C. N. Pope, and K. S. Stelle, "T duality in the Green-Schwarz formalism, and the massless / massive IIA duality map," *Nucl. Phys.* **B573** (2000) 149–176, `hep-th/9907202`.
[17] L. Wulff, "The type II superstring to order θ^4," *JHEP* **07** (2013) 123, `1304.6422`.
[18] A. A. Tseytlin, "Review of AdS/CFT Integrability, Chapter II.1: Classical AdS5xS5 string solutions," *Lett. Math. Phys.* **99** (2012) 103–125, `1012.3986`.
[19] A. Buchel, A. W. Peet, and J. Polchinski, "Gauge dual and noncommutative extension of an N=2 supergravity solution," *Phys. Rev.* **D63** (2001) 044009, `hep-th/0008076`.
[20] K. G. Wilson, "Confinement of Quarks," *Phys. Rev.* **D10** (1974) 2445–2459. [,319(1974)].
[21] J. M. Maldacena, "Wilson loops in large N field theories," *Phys. Rev. Lett.* **80** (1998) 4859–4862, `hep-th/9803002`.
[22] S.-J. Rey and J.-T. Yee, "Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity," *Eur. Phys. J.* **C22** (2001) 379–394, `hep-th/9803001`.
[23] N. Drukker and D. J. Gross, "An Exact prediction of N=4 SUSYM theory for string theory," *J. Math. Phys.* **42** (2001) 2896–2914, `hep-th/0010274`.
[24] L. F. Alday and J. M. Maldacena, "Gluon scattering amplitudes at strong coupling," *JHEP* **06** (2007) 064, `0705.0303`.
[25] A. Brandhuber, P. Heslop, and G. Travaglini, "MHV amplitudes in N=4 super Yang-Mills and Wilson loops," *Nucl. Phys.* **B794** (2008) 231–243, `0707.1153`.
[26] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, "Hexagon Wilson loop = six-gluon MHV amplitude," *Nucl. Phys.* **B815** (2009) 142–173, `0803.1466`.
[27] D. Mller, H. Mnkler, J. Plefka, J. Pollok, and K. Zarembo, "Yangian Symmetry of smooth Wilson Loops in $\mathcal{N} = 4$ super Yang-Mills Theory," *JHEP* **11** (2013) 081, `1309.1676`.
[28] H. Mnkler, "Symmetries of Maldacena-Wilson Loops from Integrable String Theory," `1712.04684`.
[29] T. Klose, F. Loebbert, and H. Munkler, "Master Symmetry for Holographic Wilson Loops," *Phys. Rev.* **D94** (2016), no. 6 066006, `1606.04104`.
[30] H. Munkler, "Bonus Symmetry for Super Wilson Loops," *J. Phys.* **A49** (2016), no. 18 185401, `1507.02474`.
[31] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *JHEP* **10** (2008) 091, `0806.1218`.
[32] M. Ammon and J. Erdmenger, *Gauge/Gravity Duality.* Cambridge University Press, 2015.

[33] M. Rangamani, "Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence," *Class. Quant. Grav.* **26** (2009) 224003, `0905.4352`.
[34] D. T. Son and A. O. Starinets, "Viscosity, Black Holes, and Quantum Field Theory," *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 95–118, `0704.0240`.
[35] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal, and U. A. Wiedemann, "Gauge/String Duality, Hot QCD and Heavy Ion Collisions," `1101.0618`.
[36] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," *Class. Quant. Grav.* **26** (2009) 224002, `0903.3246`.
[37] J. McGreevy, "Holographic duality with a view toward many-body physics," *Adv. High Energy Phys.* **2010** (2010) 723105, `0909.0518`.
[38] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity," *J. Phys.* **A42** (2009) 343001, `0904.1975`.
[39] R. Giles, "The Reconstruction of Gauge Potentials From Wilson Loops," *Phys. Rev.* **D24** (1981) 2160.
[40] N. Drukker, D. J. Gross, and H. Ooguri, "Wilson loops and minimal surfaces," *Phys. Rev.* **D60** (1999) 125006, `hep-th/9904191`.
[41] R. Blumenhagen, D. Lüst, and S. Theisen, *Basic Concepts of String Theory*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2012.
[42] D. Fursaev and D. Vassilevich, *Operators, Geometry and Quanta: Methods of Spectral Geometry in Quantum Field Theory*. Theoretical and Mathematical Physics. Springer Netherlands, 2013.
[43] D. V. Vassilevich, "Heat kernel expansion: User's manual," *Phys. Rept.* **388** (2003) 279–360, `hep-th/0306138`.
[44] J. Polchinski, *String Theory: Volume 1, An Introduction to the Bosonic String*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998.
[45] T. McLoughlin, "Review of AdS/CFT Integrability, Chapter II.2: Quantum Strings in AdS5xS5," *Lett. Math. Phys.* **99** (2012) 127–148, `1012.3987`.
[46] A. Buchel, J. G. Russo, and K. Zarembo, "Rigorous Test of Non-conformal Holography: Wilson Loops in N=2* Theory," *JHEP* **03** (2013) 062, `1301.1597`.
[47] K. Zarembo, "Supersymmetric Wilson loops," *Nucl. Phys.* **B643** (2002) 157–171, `hep-th/0205160`.
[48] N. Drukker, "1/4 BPS circular loops, unstable world-sheet instantons and the matrix model," *JHEP* **09** (2006) 004, `hep-th/0605151`.
[49] G. V. Dunne, "Functional determinants in quantum field theory," *J. Phys.* **A41** (2008) 304006, `0711.1178`.
[50] M. Mariño, *Instantons and Large N: An Introduction to Non-Perturbative Methods in Quantum Field Theory*. Cambridge University Press, 2015.
[51] I. M. Gel'fand and A. M. Yaglom, "Integration in functional spaces and its applications in quantum physics," *Journal of Mathematical Physics* **1** (1960), no. 1 48–69, `https://doi.org/10.1063/1.1703636`.
[52] J. Aguilera-Damia, A. Faraggi, L. Pando Zayas, V. Rathee, and G. A.

Silva, "Functional Determinants of Radial Operators in AdS_2," 1802.06789.
[53] R. Camporesi, "Harmonic analysis and propagators on homogeneous spaces," *Phys. Rept.* **196** (1990) 1–134.
[54] R. Camporesi, "The Spinor heat kernel in maximally symmetric spaces," *Commun. Math. Phys.* **148** (1992) 283–308.
[55] R. Camporesi and A. Higuchi, "Spectral functions and zeta functions in hyperbolic spaces," *J. Math. Phys.* **35** (1994) 4217–4246.
[56] R. Camporesi and A. Higuchi, "On the Eigen functions of the Dirac operator on spheres and real hyperbolic spaces," *J. Geom. Phys.* **20** (1996) 1–18, gr-qc/9505009.
[57] R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in AdS(5) x S**5 background," *Nucl. Phys.* **B533** (1998) 109–126, hep-th/9805028.

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