Magistère de Physique fondamentale d'Orsay

Internship's report

# Application of Casimir effect to active matter

*Intern :* Anthony BONFILS

*Supervisors :* John WETTLAUFER,
Dhrubaditya MITRA and Ralf EICHHORN

*Dates of internship : from 25/04/16 to 29/07/16*

*Institute :* NORDITA

*Roslagstullsbacken 23, 106 91 Stockholm*
*SWEDEN*

Magistère de Physique d'Orsay

# Abstract

My first task was to understand a paper written by John and not published yet on Casimir effect. After a few weeks my supervisors made explain it on the board in one hour to other researchers in statistical physics. That was a fantastic exercise and I keep an important lesson : I spontaneously tried to apply Euler-Maclaurin formula to equation 118 and mistook. I will never foreget that only doing what I am totally sure during such a presentation !
Then Dhruba made me study first chapters of Childress' book[8] with Satyajit (post-doctoral fellow working with John). This first collaboration was extremely efficient and I remember with a great pleasure our debates. After studied different derivations of Stokes' law, I determined rotational friction coefficient for a sphere by Green functions method ( see section 1.1.1 ), an exercise proposed in the book I have just studied. In the same time I began to study with Ralf Brownian motion, the theory of stochastic processes and rotational diffusion. It took me almost two monthes and required John's help to understand them properly. I also acknowledge Dhruba for helping me to clarify Itô and Stratonovich dilemma.
21st June I gave a seminar to introduce the paper[13] I had worked on with Gatien Verley and Hadrien Vroylandt during my previous internship in the Laboratoire de Physique théorique d'Orsay. That was a great honor for me to represent LPT in Nordita and a wonderful experience. I recognize it had to be a bit labourious for the audience to follow me because of my basic and very hesitating English, but I promise I will do better next time !
In the end, I mostly work with John who was absent at the beginning. He made me discover spinodal decomposition, Lifschitz theory (that I unforntunately have not understood until now), Magnus effect, Osseen problem and the method of matching asymptotic expansions in singular perturbation theory. We had long discussions on reference [26], a paper written by his friend Brady on active matter that he wanted me to explain him. He finally proposed me to come back in Nordita in one year for Phd.
All along this internship I also attent many seminars, receptions, conferences, barbecues and even a defense of Phd, without forgetting famous Monday cakes and the most important event : Nordita day !

# Résumé du stage

Ma première tâche fut de comprendre le papier de John sur l'effet Casimir (pas encore publié). Au bout de quelques semaines on me demanda de l'expliquer en une heure au tableau aux autres chercheurs en physique statistique. Cela s'avéra un formidable exercice dont je tire une importante leçon : j'ai essayé d'appliquer en direct la formule d'Euler-Maclaurin à l'équation 118 and me suis bien évidemment trompé ! Je ne suis pas prêt d'oublier qu'il est plus prudent de ne parler que de choses dont on est absolument certain au cours de telles présentations.
Ensuite Dhruba me fit étudier avec Satyajit (un postdoc de John) les premiers chapitres du livre de Childress[8]. Cette première collaboration fut extrêmement fructueuse et c'est avec un grand plaisir que je me remémore nos débats. Après avoir étudié plusieurs démonstrations de la loi de Stokes, j'ai calculé le coefficient de friction rotationelle pour une sphere avec la méthode des fonctions de Green ( cf partie 1.1.1 ), un exercice proposé dans le livre que je venais d'étudier. Dans la foulée je commençai à me pencher avec Ralf sur le mouvement brownien, la théorie des processus stochastiques et la diffusion rotationelle. Il me fallut cependant près de deux mois, ainsi que l'aide de John, pour absorber ces nouveaux concepts. Je remercie au passage Dhruba pour son aide dans la compréhension du très épineux dilemme de Itô et Stratonovich.
J'eus l'opportunité le 21 juin de donner un séminaire pour présenter la publication[13] sur laquelle j'avais travaillé avec Gatien Verley et Hadrien Vroylandt au cours de mon précédent stage au Laboratoire de Physique théorique d'Orsay. Ce fut un grand honneur pour moi de représenter le LPT à Nordita et une expérience inoubliable. Je suis forcé de reconnaître que mes auditeurs ont probablement eu quelques difficultés à me suivre en raison de mon anglais plutôt basique et très hésitant, mais je promets de faire mieux la prochaine fois !
A la fin, je travaillai principalement avec John qui était absent au début. Il me fit découvrir la décomposition spinodale, la théorie de Lifschitz (que je n'ai malheureusement toujours pas comprise à ce jour), l'effet Magnus, le problème d'Oseen et la méthode de correspondance des expansions asymptotiques en théorie des perturbations singulières. Nous eûmes de longues discussions au sujet de la publication [26] de son ami Brady sur la matière active qu'il voulait que je lui explique. Il me proposa finalement de revenir dans un an à Nordita pour faire ma thèse.
Tout au long de ce stage, j'eus l'occasion d'assister à de nombreux séminaires, réceptions, conférences, barbecues

et même à la défense d'une thèse, sans oublier les fameux gâteaux du lundi et l'évènement le plus important : Nordita day.

# Contents

# Introduction

I did this internship for the Magistère de Physique fondamentale d'Orsay in the Statistical, Biological and Condensed matter Physics' department of Nordic Institute for Theoretical Physics directed by Thors Hans Hansson in Stockholm. It is located in the AlbaNova University Center with the Astrophysics' department and the High energy Physics' department, and jointly supported by KTH, the Royal institute of Technology and Stockholm University.
My purpose was to understand Casimir effet elaborated by John and try to apply it to study active matter. I begin with report by a piece of fluids dynamics to present my calculation of rotational friction coefficient and introduce Magnus. I also give a first approach of active matter inspired by [8]. Then I mostly enter in the subject with rotational diffusion and develop a theory of active particles based on stochastic processes. In the last part, I summarize John 's paper on Casimir effect with a few comments.

# 1 A piece of fluids dynamics

## 1.1 Spinning sphere at low Reynolds number

### 1.1.1 Rotational friction coefficient

We consider now the previous sphere of radius R simply rotating with constant angular velocity $\vec{\Omega} = \Omega \vec{e}_z$ in the fluid and want to determine the torque $\vec{T}$ which is exerted on it. We look for an antisymmetric solution of Stokes equation :
$u_i = \epsilon_{ijk} \frac{\partial \chi}{\partial x_j} a_k$ with $\vec{a}$ a constant vector.
It is very useful to notice that such a spinning sphere does not modify the pressure field which is actually constant. Unfortunately I did NOT see it when I solved this exercise. That is why I simply took the $\vec{\text{curl}}$ to get rid of the gradient and proceeded as following[1] :

$$\vec{\text{curl}}\, \Delta\Big(\vec{\nabla}\chi \times \vec{a}\Big) = \vec{\text{curl}}\Big(\vec{\nabla}\Delta\chi \times \vec{a}\Big) = \vec{0} \quad \Leftrightarrow \quad \Delta^2 \chi\, \vec{a} = \big(\vec{a}\cdot\vec{\nabla}\big)\vec{\nabla}\Delta\chi \tag{1}$$

Assuming that $\Delta\chi$ is only a function of r, we write this equation in spherical coordinates (see appendix for the expression of $(\vec{a}\cdot\vec{\nabla})\vec{b}$). It only remains the radial component :

$$a_r\left(\frac{\partial^2 \Delta\chi}{\partial r^2} + \frac{2}{r}\frac{\partial \Delta\chi}{\partial r}\right) = a_r \frac{\partial^2 \Delta\chi}{\partial r^2} \quad \Rightarrow \quad \frac{\partial \Delta\chi}{\partial r} = 0 \tag{2}$$

We immediatly infer :

$$\Delta\chi = A \quad \Rightarrow \quad \chi(r) = \frac{A}{6}r^2 - \frac{B}{r} + C \tag{3}$$

Of course it would have been much easier to get this result if I had seen that actually $\Delta\vec{u} = \vec{0}$ but I must be content with technique before acquiring flair. In any case this yields :

$$u_x = \left(\frac{A}{3} + \frac{B}{r^3}\right)(y\, a_z - z\, a_y) \tag{4}$$

$$u_y = \left(\frac{A}{3} + \frac{B}{r^3}\right)(z\, a_x - x\, a_z) \tag{5}$$

$$u_z = \left(\frac{A}{3} + \frac{B}{r^3}\right)(x\, a_y - y\, a_x) \tag{6}$$

On the other side, it is to see that this solution can be written like $\vec{u} = \vec{w} \times \vec{r}$ with $\vec{w} = w(r)\vec{e}_z$ and the boundary condition $w(r = R) = \Omega$. Then we identify :

$$\begin{cases} A = C = a_x = a_y = 0 \\ B\, a_z = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} u_x = -\Omega\big(\frac{R}{r}\big)^3 y \\ u_y = \Omega\big(\frac{R}{r}\big)^3 x \\ u_z = 0 \end{cases} \tag{7}$$

[1]We use the identity $\vec{\nabla} \times (\vec{b} \times \vec{a}) = \vec{\nabla}\cdot\vec{a}\,\vec{b} - \vec{\nabla}\cdot\vec{b}\,\vec{a} + (\vec{a}\cdot\vec{\nabla})\,\vec{b} - (\vec{b}\cdot\vec{\nabla})\,\vec{a}$

At this point, the easiest way to compute the torque is to notice that the power received by the sphere is equal to the power dissipated by the fluid[17]. However the latter is exactly the Rayleigh dissipation function $\Phi$ defined in section 1.2.2. Then $\vec{T} \cdot \vec{\Omega} = \Phi$ implying :

$$T = \frac{\eta}{\Omega} \int_{V_{fluid}} \left\{ 2\left[ \left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2 \right] + \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)^2 \right\} dV \tag{8}$$

$$\Leftrightarrow \quad T = \frac{\eta}{\Omega} \int_{V_{fluid}} (x^2 + y^2) \left(\frac{dw}{dr}\right)^2 dx\, dy\, dz \tag{9}$$

$$\Leftrightarrow \quad T = \frac{4\pi\eta}{\Omega} \int_R^{+\infty} \frac{2}{3} r^4 \left(\frac{dw}{dr}\right)^2 dr \tag{10}$$

In order to write the integral in spherical coordinates, we have used a small trick :

$$\int (x^2 + y^2) f(r) dV = \int (x^2 + z^2) f(r) dV = \int (y^2 + z^2) f(r) dV \tag{11}$$

The idea is to sum these three integrals and deduce that $\int (x^2 + y^2) f(r) dV = \frac{2}{3} \int (x^2 + y^2 + z^2) f(r) dV$. Then given that $w(r) = \Omega \left(\frac{R}{r}\right)^3$, we find :

$$T = 8\pi\eta R^3 \Omega \tag{12}$$

This torque is due to viscous friction and it is opposite to rotation of the sphere :

$$\vec{T} \triangleq \oiint_{sphere} \vec{r} \times \bar{\bar{\sigma}} \cdot d\vec{S} = -8\pi\eta R^3 \vec{\Omega} \tag{13}$$

It was obtained for the first time by Kirchhoff and it is not modified when one starts from Oseen equation rather than Stokes equation.

### 1.1.2 Magnus effect

Let us imagine now our spinning sphere is also uniformly translating with velocity $\vec{U}$. Then appears a coupling between rotation and translation : this is the Magnus effect. It is famous in sports with balls because it explains unexpected curvature of trajectories, like in Roberto Carlos' hit in 1997 during the math Brasil-France. In terms of force it induces a lift sometimes called Magnus[15] force which is independent on viscosity :

$$\vec{F}_M \triangleq \pi R^3 \rho \, \vec{\Omega} \times \vec{U} \tag{14}$$

A proper derivation of this expression is given in reference [25] and uses the method of matching asymptotic expansions. Nevertheless one might guess it by having a look to Kutta-Jukowski theorem which is a computation of the lift and the drag exerted on a body moving in an inviscid fluid[16]. We presume a two dimensionnal irrotationnal steady flow and involve a large cylinder of enclosing the body with a pointing outward unit vector $\vec{n}$. The axis of the cylinder defines z-axis which is normal to flow's plane. For convenience we work in body's frame. The force we are interested in is simply the sum of stress and flux of momentum through side surface $S_S$ of the cylinder; the drag is parallel to the flow and the lift orthogonal

$$F_D = -\iint_{S_S} \left( \rho u_x \vec{u} \cdot \vec{n} - P\, n_x \right) dS \tag{15}$$

$$F_L = -\iint_{S_S} \left( \rho u_y \vec{u} \cdot \vec{n} - P\, n_y \right) dS \tag{16}$$

Bernoulli theorem provides pressure field : $P + \frac{1}{2}\rho u^2 = P_0 + \frac{1}{2}\rho U^2$.
Then writting integrals in polar coordinates, introducing a small disturbance $\vec{u}_1$ such that $\vec{u} = -\vec{U} + \vec{u}_1$ and

discarding second order terms, it is easy to find after a few calculations :

$$F_D = \rho U \iint_{S_S} \vec{u}_1 \cdot \vec{n} dS \equiv \iiint_{cylinder} \operatorname{div} \vec{u}_1 dV = 0 \quad \text{by incompressibility} \tag{17}$$

$$F_L = \rho U\, l_z \oint_{\mathscr{C}} \vec{u}_1 \cdot \vec{dl} \equiv \rho U\, l_z \oint_{\mathscr{C}} \vec{u} \cdot \vec{dl} \tag{18}$$

where $l_z$ is the length of the cylinder and $\mathscr{C}$ the curve drawn by its cross section. Introducing the circulation vector $\vec{\Gamma} \triangleq \oint_{\mathscr{C}} \vec{u} \cdot \vec{dl}\, \vec{e}_z$, as usual in hydrodynamics, we immediatly infer :

$$\vec{F}_L = l_z \rho \vec{\Gamma} \times \vec{U} \tag{19}$$

This lift is very similar to Magnus force even though this derivation is valid only for inviscid fluids. Besides we have just obtained a drag equal to zero : it is the paradox of D'Alembert.[22]

## 1.2 A first approach of active matter[8]

### 1.2.1 Motion of a swimmer

Let us consider a big fixed volume $V_{tot}$ delimited by the surface $\Sigma$ and inside it a swimmer of volume $V_1(t)$ and surface S(t) can move. We are interested in the derivative with respect to time of an extensive time and space-dependent amount $\boldsymbol{Q}$ on the volume surrounding the swimmer $V(t) \triangleq V_{tot} - V_1(t)$ :

$$\frac{d}{dt}\left\{ \int_{V(t)} \boldsymbol{Q} dV \right\} = \frac{d}{dt}\left\{ \int_{V_{tot}} \boldsymbol{Q} dV_{tot} \right\} - \frac{d}{dt}\left\{ \int_{V_1(t)} \boldsymbol{Q} dV_1 \right\} \equiv \int_{V_{tot}} \frac{\partial \boldsymbol{Q}}{\partial t} dV_{tot} - \int_{V_1(t)} \frac{\partial \boldsymbol{Q}}{\partial t} dV_1 - \int_{S(t)} \boldsymbol{Q}\, \vec{\dot{r}}_S \cdot \vec{n} dS \tag{20}$$

We have exchanged integration and time derivative on $V_{tot}$ given that it is fixed, and applyed the Reynolds transport theorem to $V_1(t)$ with $\vec{\dot{r}}_S$ the velocity of the surface S(t). Then we recognize an integral on V(t) to get the following useful relation :

$$\frac{d}{dt}\left\{ \int_{V(t)} \boldsymbol{Q} dV \right\} = \int_{V(t)} \frac{\partial \boldsymbol{Q}}{\partial t} dV - \int_{S(t)} \boldsymbol{Q}\, \vec{\dot{r}}_S \cdot \vec{n} dS \tag{21}$$

When $\boldsymbol{Q}$ is the momentum, it gives for an incompressible fluid :

$$\frac{d}{dt}\left\{ \int_{V(t)} \rho \vec{u} dV \right\} = \int_{V(t)} \rho \frac{\partial \vec{u}}{\partial t} dV - \int_{S(t)} \rho \vec{u}\, \vec{\dot{r}}_S \cdot \vec{n} dS \tag{22}$$

The rate of change of momentum inside V(t) is equal to the sum of the inward flux of momentum and the sources of momentum ie the forces exerted on V(t) ; we define the pressure such that it compensates gravity (neutral boyancy). Noting that $\vec{n}$ is outward $\Sigma$ but inward S(t), we have :

$$\int_{V(t)} \rho \frac{\partial \vec{u}}{\partial t} dV = \int_{\Sigma} \Big( - \rho \vec{u}\, \vec{u} \cdot \vec{n} + \bar{\bar{\sigma}} \cdot \vec{n} \Big) d\Sigma + \int_{S(t)} \Big( \rho \vec{u}\, \vec{u} \cdot \vec{n} - \bar{\bar{\sigma}} \cdot \vec{n} \Big) dS \tag{23}$$

With the boundary condition $\vec{u} = \vec{\dot{r}}_S$ at the surface of the swimmer and using equation 23, equation 22 becomes :

$$\frac{d}{dt}\left\{ \int_{V(t)} \rho \vec{u} dV \right\} = \int_{\Sigma} \Big( - \rho \vec{u}\, \vec{u} \cdot \vec{n} + \bar{\bar{\sigma}} \cdot \vec{n} \Big) d\Sigma + \vec{F} \tag{24}$$

where $\vec{F} \triangleq - \int_{S(t)} \bar{\bar{\sigma}}.\vec{n} dS$ is the force exerted by the swimmer on the fluid and it is equal to zero on average for a free locomotion according to the first Newton's law. Hence, taking the time average of equation24 :

$$\left\langle \int_{\Sigma} \Big( \rho \vec{u}\, \vec{u} \cdot \vec{n} - \bar{\bar{\sigma}} \cdot \vec{n} \Big) d\Sigma \right\rangle = \vec{0} \tag{25}$$

Therefore in a free motion, self-propulsion of the swimmer exactly compensates in average its drag. In other words : when the swimmer pushes back on the fluid it creates a flux of momentum $\int_{\Sigma} \rho \vec{u}\, \vec{u} \cdot \vec{n} d\Sigma$ but there is also a resistance $- \int_{\Sigma} \bar{\bar{\sigma}} \cdot \vec{n} d\Sigma$ because of viscosity. The mean motion is uniform so that the sum is equal to zero.

### 1.2.2 Conservation of energy

The Navier-Stokes equation is :

$$\rho \frac{d\vec{u}}{dt} = -\vec{\nabla} P + \eta \Delta \vec{u} + \rho \vec{g} \tag{26}$$

We do a dot product with the speed :

$$\rho \vec{u} \cdot \left( \frac{\partial \vec{u}}{\partial t} + \frac{1}{2} \vec{\nabla} u^2 + \vec{\text{curl}}\, \vec{u} \times \vec{u} \right) = \vec{u} \cdot \vec{\nabla} \cdot \bar{\bar{\sigma}} + \rho \vec{u} \cdot \vec{g} \tag{27}$$

But the mixed product disappears and using the identity[2] $\vec{\nabla} \cdot \left( \bar{\bar{\sigma}} \cdot \vec{u} \right) = \vec{u} \cdot \vec{\nabla} \cdot \bar{\bar{\sigma}} + \vec{\nabla} \vec{u} : \bar{\bar{\sigma}}$, so that equation 27 becomes :

$$\frac{1}{2} \rho \left( \frac{\partial u^2}{\partial t} + \vec{u} \cdot \vec{\nabla} u^2 \right) = \vec{\nabla} \cdot \left( \bar{\bar{\sigma}} \cdot \vec{u} \right) - \vec{\nabla} \vec{u} : \bar{\bar{\sigma}} + \rho \vec{u} \cdot \vec{g} \tag{28}$$

At this point, it is interesting to notice that $\vec{u} \cdot \vec{\nabla} u^2 = \vec{\nabla} \cdot \left( u^2 \vec{u} \right) - u^2 \vec{\nabla} \cdot \vec{u}$ with $\text{div}\, \vec{u} = 0$ by incompressibility. Then when we integrate on V(t), the divergence theorem ensures :

$$\int_{V(t)} \frac{1}{2} \rho \frac{\partial u^2}{\partial t} dV + \int_{\Sigma} \left( \frac{1}{2} \rho u^2 \vec{u} \cdot \vec{n} - (\bar{\bar{\sigma}} \cdot \vec{u}) \cdot \vec{n} \right) d\Sigma - \int_{S(t)} \left( \frac{1}{2} \rho u^2 \vec{u} \cdot \vec{n} - (\bar{\bar{\sigma}} \cdot \vec{u}) \cdot \vec{n} \right) dS = \int_{V(t)} \left( \rho \vec{u} \cdot \vec{g} - \vec{\nabla} \vec{u} : \bar{\bar{\sigma}} \right) dV \tag{29}$$

Let $W_\Sigma \triangleq \int_\Sigma \vec{u} \cdot \bar{\bar{\sigma}} \cdot \vec{n} d\Sigma$ and $W_S \triangleq -\int_{S(t)} \vec{u} \cdot \bar{\bar{\sigma}} \cdot \vec{n} dS$ be the powers of the forces respectively exerted on surfaces $\Sigma$ and S(t). We also call $\dot{E} \triangleq \frac{d}{dt} \left\{ \int_{V(t)} \frac{1}{2} \rho u^2 dV \right\}$ the rate of change of kinetic energy inside the volume of fluid V(t) and recognize $G \triangleq \int_{V(t)} \rho \vec{u} \cdot \vec{g} dV$ the power of the gravitationnal force. With these notations we rewrite equation 29 :

$$\int_{\Sigma} \frac{1}{2} \rho u^2 \vec{u} \cdot \vec{n} d\Sigma + \int_{V(t)} \frac{\partial \left( \frac{1}{2} \rho u^2 \right)}{\partial t} dV - \int_S \frac{1}{2} \rho u^2 \vec{u} \cdot \vec{n} dS + \int_{V(t)} \eta \frac{\partial u_i}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dV = W_\Sigma + W_S + G + \int_{V(t)} P \underbrace{\frac{\partial u_i}{\partial x_i}}_{=0} dV \tag{30}$$

We define the Rayleigh dissipation function by :

$$\Phi \triangleq \int_{V(t)} \eta \frac{\partial u_i}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dV = \int_{V(t)} \frac{1}{2} \eta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dV \equiv \int_{V(t)} 2\eta\, \bar{\bar{\tau}} : \bar{\bar{\tau}}^\dagger dV \tag{31}$$

where $\bar{\bar{\tau}}$ is the symmetric part of tensor gradient of speed ie the strain tensor. On the other hand, we use equation 21 with $= \frac{1}{2} \rho u^2$ in order to eventually get an equation of energy conservation :

$$\dot{E} + J_\Sigma + \Phi = W_S + W_\Sigma + G \tag{32}$$

where $J_\Sigma \triangleq \int_\Sigma \frac{1}{2} \rho u^2 \vec{u} \cdot \vec{n} d\Sigma$ is the outward flux of kinetic energy through $\Sigma$. In this equation, $\Phi$ stands for the rate of production of heat due to viscous dissipation. Thus it is actually the first principle of thermodynamics :

power of forces = rate of energy change (only kinetic here) + rate of heat output

As previously we define pressure to compensate gravity and also make $\Sigma$ expands to infinity so that it only remains :

$$\dot{E} + \Phi = W_S \quad \Rightarrow \quad \langle \Phi \rangle = \langle W_S \rangle \tag{33}$$

We have just found back what we showed in the end of section 1.2.1 : in time average viscous force exactly compensates self-propulsion. Indeed according to Newton's third law forces exerted on the swimmer are equal to forces exerted by the swimmer on the fluid to propel itself, then $\langle W_S \rangle$ is also the mean power of self-propulsion. We are finally able to define of swimming by $\bar{\eta} \triangleq \frac{UT}{\langle \Phi \rangle}$, with T the thrust of the swimmer and U its mean speed. Rigorously we also should take into account dissipation due to metabolism of the swimmer.

[2] $\frac{\partial (\sigma_{ij} u_j)}{\partial x_i} = u_j \frac{\partial \sigma_{ij}}{\partial x_i} + \frac{\partial u_j}{\partial x_i} \sigma_{ij}$

# 2 Active matter

Everybody have ever admired flocks of birds in the sky and schools of fishes in the sea, or followed the motion of a swarm of beas, a herd of cows or spectators in a concert of heavy metal. And you might be astonished by the subtile combination of disorder and organization of these motions. Indeed even though each creature has a very complicated individual trajectory, movement of groups is relatively simple. These are examples of active matter and playing fields for statistical physicists. For a few years very intensive studies, as theoretical as experimental, have been being developed to explain those macroscopic collective behaviours emerging from microscopic agitation ; see [24], [27] and their references for a more detailed introduction to the topic and a good summary of results obtained in the past decade.
We studied the motion of one swimmer in a fluid in section 1.2 and especially its interaction with the fluid. Now we are interested in the influence of other swimmers using the theory of stochastic processes. Before elaborating a theory of active particles, we give an introduction to rotational diffusion. We will compute the stress of active particles and finally discuss possible extensions of our theory.

## 2.1 Rotational diffusion

### 2.1.1 Formal approach[10]

Imagine a Brownian motion in a unit sphere. A particle on this sphere is repered by a unit vector $\vec{n}(t)$ and it has no degree of freedom for translation ; it is only able to move on the sphere. So this is a rotational Brownian motion characterized not by a random force but by a random torque.
The infinitesimal variation $\partial\vec{n}$ is tangent to the sphere whereas $\vec{n}$ is normal. Then this variation is equal to the gradient mines its orthogonal projection over $\vec{n}$ :

$$\frac{\partial}{\partial\vec{n}} = \big(\mathbb{1} - \vec{n}\otimes\vec{n}\big)\vec{\nabla} \tag{34}$$

It is useful to compute the square ; calculations are a bit long but easy[3].

$$\frac{\partial^2}{\partial\vec{n}^2} = (\delta_{ij} - n_i n_j)\partial_j(\delta_{ik} - n_i n_k)\partial_k = \partial_i\partial_i - n_i n_j\partial_i\partial_j - 2n_i\partial_i \equiv \Delta - \big(\vec{n}\cdot\vec{\nabla}\big)^2 - \vec{n}\cdot\vec{\nabla} \tag{35}$$

According to Varignon theorem, $\frac{d\vec{n}}{dt} = \vec{\Omega}(t)\times\vec{n}(t)$ and in our case $\vec{\Omega}(t)$ is a Gaussian white noise :

$$\left\langle\vec{\Omega}(t)\right\rangle = \vec{0} \tag{36}$$

$$\langle\Omega_i(0)\Omega_j(t)\rangle = 2D_r\delta_{ij}\delta(t) \tag{37}$$

We want to determine a Fokker-Planck equation for the probability distribution of $\vec{n}(t)$. For mathematical convenience we define an operator $\mathcal{O}$ and a function $\vec{f}$ as following :

$$\frac{d\vec{n}}{dt} = \vec{f}\big(\vec{n}(t),t\big) = \mathcal{O}(t)\vec{n}(t) \quad\Rightarrow\quad \begin{cases}\mathcal{O}(t)\equiv\frac{d\vec{f}}{d\vec{n}}\\ \langle\mathcal{O}(t)\rangle = 0\\ \langle\mathcal{O}_{ik}(0)\mathcal{O}_{ln}(t)\rangle = \langle\epsilon_{ijk}\Omega_j(0)\epsilon_{lmn}\Omega_m(t)\rangle \equiv 2D_r(\delta_{il}\delta_{kn} - \delta_{in}\delta_{kl})\delta(t)\end{cases} \tag{38}$$

We are going to use the method given in reference ... and compute the moments defined by :

$$a_i^{(1)} \triangleq \lim_{\delta t\to 0}\left\langle\frac{\delta n_i}{\delta t}\right\rangle \tag{39}$$

$$a_{ij}^{(2)} \triangleq \lim_{\delta t\to 0}\left\langle\frac{\delta n_i\delta n_j}{\delta t}\right\rangle \tag{40}$$

[3] NB : $(\vec{n}\cdot\vec{\nabla})^2 = n_i\partial_i(n_j\partial_j) = n_i\partial_i + n_i n_j\partial_i\partial_j$ since $\partial_i n_j = \delta_{ij}$

This calculation is a bit complicated but mathematically interesting.

$$\delta\vec{n} \triangleq \int_0^{\delta t} dt' \vec{f}\big(\vec{n}(t'),t'\big) = \int_0^{\delta t} dt' \vec{f}\Big(\int_0^{t'} dt'' \vec{f}\big(\vec{n}(t''),t''\big),t'\Big) \simeq \int_0^{\delta t} dt' \vec{f}\big(\vec{n}_0,t'\big) + \int_0^{\delta t} dt' \frac{d\vec{f}}{d\vec{n}}\Big|_{\vec{n}_0,t'} \int_0^{t'} dt'' \vec{f}\big(\vec{n}_0,t''\big) + O(\delta t^2) \tag{41}$$

Then we infer :

$$\langle \delta n_i \rangle = \int_0^{\delta t} dt' \big\langle \mathcal{O}_{ij}(t') \big\rangle n_j + \int_0^{\delta t} dt' \int_0^{t'} dt'' \big\langle \mathcal{O}_{ik}(t')\mathcal{O}_{kj}(t'') \big\rangle n_j + O(\delta t^2) = -2D_r \, \delta t \, n_i + O(\delta t^2) \tag{42}$$

$$\langle \delta n_i \delta n_j \rangle = \int_0^{\delta t} dt' \int_0^{\delta t} dt'' \big\langle \mathcal{O}_{ik}(t')\mathcal{O}_{jl}(t'') \big\rangle n_k n_l + O(\delta t^2) = 2D_r \, \delta t \, (\delta_{ij}||n||^2 - n_i n_j) + O(\delta t^2) \tag{43}$$

We used the important identity $\int_0^t dt' \delta(t-t') = \frac{1}{2}$.
At this stage do NOT replace $||n||^2$ by 1. Indeed the norm is conserved but its value is not specifyed. We finally write the Fokker-Planck equation after a bit fastidious calculations :

$$\frac{\partial P}{\partial t} = -\partial_i\big[a_i^{(1)} P\big] + \frac{1}{2}\partial_i\partial_j\big[a_{ij}^{(2)} P\big] \quad \Leftrightarrow \quad \frac{\partial P}{\partial t} = D_r\big(\partial_i\partial_i - n_i n_j \partial_i \partial_j - 2 n_i \partial_i\big) P \tag{44}$$

Then thanks to equation 35 we put it in an easy form :

$$\frac{\partial P}{\partial t} = D_r \frac{\partial^2 P}{\partial \vec{n}^2} \tag{45}$$

$D_r$ is the rotational diffusion coefficient. We are going to compute it in the next section and also add a drift term.

### 2.1.2 Einstein's relation : a fluctuation-dissipation theorem[20]

Let us consider the rotation of a body in a fluid such that it is submitted to a random torque $\vec{T}(t)$. Orbital momentum is related to angular velocity thanks to tensor of inertia : $\vec{J} = \bar{\bar{I}} \cdot \vec{\Omega}$. For simple geometry this tensor is a scalar. We call $\gamma_r$ the friction coefficient ; we have shown in section 1.1.1 that $\gamma_r = 8\pi\eta R^3$ for a sphere. Theorem of orbital momentum provides :

$$I\frac{d\vec{\Omega}}{dt} = -\gamma_r \vec{\Omega}(t) + \vec{T}(t) \quad \Rightarrow \quad \vec{\Omega}(t) = \vec{\Omega}(0) e^{-\frac{\gamma_r}{I}t} + \frac{1}{I}\int_0^t dt' \vec{T}(t') e^{-\frac{\gamma_r}{I}(t-t')} \tag{46}$$

The torque is a Gaussian white noise : $\Big\langle \vec{T}(t) \Big\rangle = \vec{0}$ and $\Big\langle \vec{T(t)} \otimes \vec{T(t')} \Big\rangle = \bar{\bar{G}}\delta(t-t')$. Then :

$$\lim_{t\to+\infty} \Big\langle \vec{\Omega(t)} \otimes \vec{\Omega(t)} \Big\rangle = \frac{1}{2I\gamma_r}\bar{\bar{G}} \tag{47}$$

On the other side, from König theorem we have $\langle E_{kin}^{rot} \rangle = \frac{1}{2} I \left\langle \Omega^2 \right\rangle$ and from equipartition theorem $\langle E_{kin}^{rot} \rangle = \alpha \frac{k_B T}{2}$ with $\alpha$ the number of degrees of freedom. Thus

$$\lim_{t\to+\infty} \Big\langle \vec{\Omega(t)} \otimes \vec{\Omega(t)} \Big\rangle = \frac{\left\langle \Omega^2 \right\rangle}{\alpha} \mathbb{1} \equiv \frac{k_B T}{I} \mathbb{1} \tag{48}$$

let us identify $\bar{\bar{G}} = 2\gamma_r k_B T \mathbb{1}$. Furthermore given that for long times $\vec{\Omega} = \frac{1}{\gamma_r}\vec{T}$ and using equation 37, we finally obtain Einstein's relation :

$$D_r = \frac{k_B T}{\gamma_r} \tag{49}$$

This relation is sometimes called fluctuation-dissipation theorem, even though "fluctuation-dissipation theorem" rather refers to relation between correlation function and response function in linear response theory.

### 2.1.3 Drift term[9]

We can follow the rotation of the body with the unit vector $\vec{n}$ defined in section 2.1.1. We need an expression for the random torque, so we consider an infinitesimal rotation of $\delta\theta$ around $\vec{e}_z$. The work is the opposite of the change of potential energy :

$$-\vec{T}\cdot\vec{e}_z\delta\theta = U(\vec{n}+\vec{e}_z\times\vec{n}\delta\theta) - U(\vec{n}) \simeq (\vec{e}_z\times\vec{n})\cdot\vec{\nabla}U\delta\theta = \vec{e}_z\cdot\left(\vec{n}\times\vec{\nabla}U\right)\delta\theta \tag{50}$$

We identify $\vec{T} = \vec{n}\times\frac{\partial U}{\partial\vec{n}}$. It is possible to show we have to add a "Brownian potential" $k_BT\ln P$ so as to get the correct Fokker-Planck equation.

If the flow is not potential, the fluid can induce an extra angular velocity $\vec{\Omega}_0$. So we replace the gradient by $(\vec{u}\cdot\vec{\nabla})\vec{n} = \frac{\partial u_i}{\partial x_j}n_j\vec{e}_i \equiv \bar{\bar{\kappa}}\cdot\vec{n}$ with $\bar{\bar{\kappa}}$ the velocity gradient :

$$\frac{d\vec{n}}{dt} = \left(\mathbb{1}-\vec{n}\otimes\vec{n}\right)\cdot\bar{\bar{\kappa}}\cdot\vec{n} = \bar{\bar{\kappa}}\cdot\vec{n} - (\vec{n}\otimes\vec{n}:\bar{\bar{\kappa}})\vec{n} \tag{51}$$

On the other side :

$$\frac{d\vec{n}}{dt} = \vec{\Omega}_0\times\vec{n} \quad\Rightarrow\quad \vec{\Omega}_0 = \vec{n}\times\frac{d\vec{n}}{dt} \equiv \vec{n}\times\bar{\bar{\kappa}}\cdot\vec{n} \tag{52}$$

Hence

$$\vec{\Omega} = \vec{n}\times\bar{\bar{\kappa}}\cdot\vec{n} - \frac{1}{\gamma_r}\left(\vec{n}\times\vec{\nabla}\right)(U + k_BT\ln P) \tag{53}$$

Fokker-Planck equation is simply the conservation equation of P :

$$\frac{\partial P}{\partial t} = -\vec{\nabla}\cdot\left(\vec{\Omega}\times\vec{n}P\right) = -\left(\vec{n}\times\vec{\nabla}\right)\cdot(\vec{\Omega}P) \tag{54}$$

Then plugging equation 53 into it, we finally get :

$$\frac{\partial P}{\partial t} = \frac{k_BT}{\gamma_r}\left(\vec{n}\times\vec{\nabla}\right)^2 P + \left(\vec{n}\times\vec{\nabla}\right)\cdot\left(\frac{P}{\gamma_r}\left(\vec{n}\times\vec{\nabla}\right)U\right) - \left(\vec{n}\times\vec{\nabla}\right)\cdot\left(\vec{n}\times\bar{\bar{\kappa}}\cdot\vec{n}P\right) \tag{55}$$

It is easy to show that $\left(\vec{n}\times\vec{\nabla}\right)^2 \equiv \frac{\partial^2}{\partial\vec{n}^2}$ and then find back Einstein's relation on the right hand side.

## 2.2 Active particles

A so-called active particle is characterized by its ability to draw kinetic energy from its environment which confers it an extra degree of freedom. In other words an active particle is a self-propelled particle and we are going to show that it actually corresponds to a correlated Brownian motion. It is obviously a Brownian particle since it keeps hitting other particles and continuously changes its orientation. We often work with spheres because it is easy and we have an expression for friction coefficients, but in reality active particles have a more complicated shape. We can consider that orientation, and then direction of motion, is determined by the " head " given that a great number of active particles are living creatures, even though bubbles in Champagne and soda also enter in this class. However active particles are more than Brownian particles because of their self-propulsion. That is why we add a term in Langevin equation : we call $v_0$ the velocity related to the property of activity and $\vec{n}(t)$ the unit vector defining orientation of the particle, then a good model for self-propulsive force is[26] $\vec{F}_{swim} \triangleq -\gamma v_0\vec{n}(t)$ with $\gamma$ the translational friction coefficient given by Stokes' law.

### 2.2.1 A correlated Brownian motion[28]

The unit vector $\vec{n}(t)$ is submitted to rotational diffusion, therefore we have to introduce two Gaussian white noises : $\vec{\xi}$ for translation and $\vec{\xi}_r$ for rotation. We remind that :

$$\left\langle \vec{\xi} \right\rangle = \left\langle \vec{\xi}_r \right\rangle = \vec{0} \tag{56}$$

$$\left\langle \vec{\xi}(t) \otimes \vec{\xi}(t') \right\rangle = 2D\delta(t-t')\mathbb{1} \quad \text{with} \quad D = \frac{k_B T}{\gamma} \tag{57}$$

$$\left\langle \vec{\xi}_r(t) \otimes \vec{\xi}_r(t') \right\rangle = 2D_r\delta(t-t')\mathbb{1} \quad \text{with} \quad D_r = \frac{k_B T}{\gamma_r} \tag{58}$$

The couple of Langevin describing the overdamped motion of a Brownian active particle is :

$$\begin{cases} \frac{d\vec{r}}{dt} = v_0\vec{n}(t) - \frac{1}{\gamma}\vec{\nabla}U + \vec{\xi} \\ \frac{d\vec{n}}{dt} = \vec{\xi}_r \times \vec{n}(t) \end{cases} \tag{59}$$

where U is the potential of interaction of particules, hard spheres potential for instance. This couple is actually equivalent to the following :

$$\frac{d\vec{r}}{dt} = -\frac{1}{\gamma}\vec{\nabla}U + \vec{\xi} + \vec{\chi} \quad \text{with} \quad \langle\vec{\chi}\rangle = \vec{0} \quad \text{and} \quad \left\langle \vec{\chi}(t) \otimes \vec{\chi}(t') \right\rangle = \frac{v_0^2}{3}e^{-2D_r|t-t'|}\mathbb{1} \tag{60}$$

The derivation provided by Farage and al in reference ... is so beautiful that it would be a pity of not to give some details about it in such a report.

We start with the Fokker-Planck equation for rotational diffusion. For simplicity we have to skip the drift term, then the probability density of transition checks :

$$\frac{\partial f(\vec{n},t|\vec{n}_0,t_0)}{\partial t} = D_r\frac{\partial^2 f(\vec{n},t|\vec{n}_0,t_0)}{\partial \vec{n}^2} \tag{61}$$

In spherical coordinates $\vec{n}\cdot\vec{\nabla} = r\partial_r$ with $r=1$, so it is easy to show that $\frac{\partial^2}{\partial\vec{n}^2} \equiv \Delta - \partial_r^2 - 2\partial_r$ which is the angular part of Laplacian. It is also very interesting to notice that $\frac{\partial^2}{\partial\vec{n}^2} \equiv -\vec{\mathcal{L}}^2$ with $\vec{\mathcal{L}}$ the quantum angular momentum. This makes us expand the solution in spherical harmonics :

$$f(\theta,\varphi,t|\theta_0,\varphi_0,t_0) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l} A_{lm}(t|\theta_0,\varphi_0,t_0)Y_{lm}(\theta,\varphi) \tag{62}$$

Then using $\vec{\mathcal{L}}^2 Y_{lm} = l(l+1)Y_{lm}$, equation 61 becomes :

$$\sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\frac{\partial A_{lm}}{\partial t}Y_{lm} = -D_r\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} l(l+1)A_{lm}Y_{lm} \tag{63}$$

We multiply by $Y^\star_{l'm'}(\theta,\varphi)$ and integrate with $\int d\Omega\, Y^\star_{l'm'}Y_{lm} = \delta_{mm'}\delta_{ll'}$ to find :

$$\frac{\partial A_{lm}}{\partial t} = -D_r l(l+1)A_{lm} \quad \Rightarrow \quad A_{lm}(t|\theta_0,\varphi_0,t_0) = a_{lm}(\theta_0,\varphi_0)\,e^{-D_r l(l+1)(t-t_0)} \tag{64}$$

Let us rewrite the initial condition :

$$f(\theta,\varphi,t_0|\theta_0,\varphi_0,t_0) = \delta(\theta-\theta_0)\delta(\varphi-\varphi_0) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)Y^\star_{lm}(\theta_0,\varphi_0) \equiv \sum_{l=0}^{+\infty}\sum_{m=-l}^{l} a_{lm}(\theta_0,\varphi_0)Y_{lm}(\theta,\varphi) \tag{65}$$

We identify $a_{lm}(\theta_0,\varphi_0) \equiv Y^\star_{lm}(\theta_0,\varphi_0)$ and conclude :

$$f(\theta,\varphi,t|\theta_0,\varphi_0,t_0) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi)Y^\star_{lm}(\theta_0,\varphi_0)e^{-D_r l(l+1)(t-t_0)} \tag{66}$$

When t goes to infinity only terms with l=0 survive. We immediately infer

$$\lim_{t\to+\infty} f(\theta,\varphi,t|\theta_0,\varphi_0,t_0) = \frac{1}{4\pi} \triangleq f_{eq} \tag{67}$$

We eventually define a new stochastic process : $\vec{\chi} \triangleq v_0\vec{n}$. Using properties of spherical harmonics[4] and the following definitions

$$\langle n_i\rangle = \int d\Omega\, f_{eq}\, n_i(\theta,\varphi) \tag{68}$$

$$\langle n_i(t)n_j(t_0)\rangle = \int d\Omega\, d\Omega_0 f_{eq}\, n_i(\theta,\varphi)\, n_j(\theta_0,\varphi_0) f(\theta,\varphi,t|\theta_0,\varphi_0,t_0) \tag{69}$$

we get equation 60.
$\vec{\chi}$ is not a white noise. Its correlation involves a decreasing exponential which implies a non trivial fluctuation spectrum. That is why we talk about "correlated" Brownian motion.

### 2.2.2 Direction correlation function

Let us consider a system of active particles. The motion of one particle is modeled as following[19] :
- The trajectory is a succession of runs in straight line characterized by a speed direction and a time duration. Tumbles occur at discrete times $t_i$ and the new direction is governed by an azimuthally symmetric probability distribution with respect the former. In addition the angle between is the former and the new direction is independent on the previous run.
- We assume that all runs have the same constant speed v.
- For a given direction the time duration of a run is governed by an exponential law ie a without aging law. Indeed the probability that a tumble occurs during a run (and therefore closes it) is independent on the past of the particle.
- We presume that tumbles are Poisson distributed : the probability density that n tumbles occur during a trajectory [0,t] is $P_n(t) = (\frac{t}{\mathcal{T}})^n \frac{e^{-\frac{t}{\mathcal{T}}}}{n!}$ with $\mathcal{T}$ the mean time duration of a run.

The instantaneous direction of the motion is characterized by a time dependent unit vector $\vec{a}(t)$ and we define the direction correlation function by $C(t) \triangleq \langle\vec{a}(0)\cdot\vec{a}(t)\rangle$, where $\langle...\rangle$ is the average over all initial directions and all possible subsequent trajectories. Let us suppose that n tumbles occurs between 0 and t with $\alpha_i \triangleq \vec{a}(t_{i-1})\cdot\vec{a}(t_i)$ with $t_0 = 0$ and $t_{n+1} = t$. In order to compute C(t), it is possible to see the trajectory as a polymer chain with free rotation around bounds[11]. In this case, $\langle...\rangle$ stands for the average over all configurations. The assumption of free rotation implies that the mean projection in transverse direction is zero ; this is denoted by the subscript $\perp$.

$$\langle\vec{a}(0)\cdot\vec{a}(t)\rangle_\perp = \big(\vec{a}(t_0)\cdot\vec{a}(t_1)\big)\big(\vec{a}(t_1)\cdot\vec{a}(t_2)\big)...\big(\vec{a}(t_{n-1})\cdot\vec{a}(t_n)\big) = \prod_{i=1}^{n}\alpha_i \tag{70}$$

Let $\alpha$ be the mean cosine of the angle between successive trajectories and add the subscript $\theta$ to denote this average. This yields :

$$\langle\vec{a}(0)\cdot\vec{a}(t)\rangle_{\perp,\theta} = \alpha^n \tag{71}$$

We eventually average over the number of tumbles which occur between 0 and t :

$$C(t) = \sum_{n=0}^{\infty}\alpha^n P_n(t) = e^{-\frac{t}{\mathcal{T}}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha t}{\mathcal{T}}\right)^n \equiv e^{-\frac{t}{\tau_c}} \tag{72}$$

4

$$Y_{10} = Y_{10}^{\star} = \sqrt{\frac{3}{4\pi}}\cos\theta$$
$$\sin\theta\,\cos\varphi = \sqrt{\frac{2\pi}{3}}\Big(Y_{11}+Y_{1-1}\Big) = \sqrt{\frac{2\pi}{3}}\Big(Y_{11}^{\star}+Y_{1-1}^{\star}\Big)$$
$$i\sin\theta\,\sin\varphi = \sqrt{\frac{2\pi}{3}}\Big(Y_{11}-Y_{1-1}\Big) = \sqrt{\frac{2\pi}{3}}\Big(Y_{1-1}^{\star}-Y_{11}^{\star}\Big)$$

with the correlation time defined by $\tau_c \triangleq \frac{\mathcal{T}}{1-\alpha}$.

### 2.2.3 Rotational AND translational diffusion

Previously we have only considered the translational diffusion that will be denoted by the subscript t. Nevertheless, in reality the runs are not straight lines ; for instance Berg and Brown[6] showed that the direction of the bacteria *Escherichia Coli* changes average about 27° during a run. This is due to the rotational diffusion[5], that we will denote by the subscript r. In order to take it into account we have to refine our model. A run of mean time duration $T_t$[5] might be seen also as a succession of much shorter runs in straight line. We apply the same assumptions than for translational diffusion to describe it and qualify by turns the infinitesimal change of direction between two tumbles. We apply the averaging procedure in decomposing the trajectory as a succession of infinitesimal runs of mean time duration $T_r$ separated by turns and on much larger time scale $T_t$ occur tumbles. Let us suppose that there are n tumbles between 0 and t, and that there are $m_i$ turns between the tumbles i and i+1. Turns occur at times $t_{ij_i}$ with $j_i \in [\![1, m_i]\!]$ and $i \in [\![0, n]\!]$ denoting the tumbles occurring at times $t^-_{ij_{m_i}+1}$ ; note the correspondence $t^+_{ij_{m_i}+1} = t_{i0}, t_{00} = 0$ and $t_{nm_{j_n}+1} = t$. If we define $\alpha^r_{ij_i} \triangleq \vec{a}(t_{ij_i-1}) \cdot \vec{a}(t_{ij_i})$ and $\alpha^t_i \triangleq \vec{a}(t_{i-1m_{i-1}+1}) \cdot \vec{a}(t_{i0})$, equation 70 becomes :

$$\langle \vec{a}(0) \cdot \vec{a}(t) \rangle_\perp = \Big( \Big(\vec{a}(t_{00}) \cdot \vec{a}(t_{01})\Big)\Big(\vec{a}(t_{01}) \cdot \vec{a}(t_{02})\Big) ... \Big(\vec{a}(t_{0m_0-1}) \cdot \vec{a}(t_{0m_0})\Big)\Big) \Big(\vec{a}(t_{0m_0+1}) \cdot \vec{a}(t_{10})\Big)$$
$$\Big( \Big(\vec{a}(t_{10}) \cdot \vec{a}(t_{11})\Big)\Big(\vec{a}(t_{11}) \cdot \vec{a}(t_{12})\Big) ... \Big(\vec{a}(t_{1m_1-1}) \cdot \vec{a}(t_{1m_1})\Big)\Big) \Big(\vec{a}(t_{1m_1+1}) \cdot \vec{a}(t_{20})\Big) ... \Big(\vec{a}(t_{n-1m_{n-1}+1}) \cdot \vec{a}(t_{n0})\Big)$$
$$\Big( \Big(\vec{a}(t_{n0}) \cdot \vec{a}(t_{n1})\Big)\Big(\vec{a}(t_{n1}) \cdot \vec{a}(t_{n2})\Big) ... \Big(\vec{a}(t_{nm_n-1}) \cdot \vec{a}(t_{nm_n})\Big)\Big) \equiv \prod_{j_0=1}^{m_0} \alpha^r_{0j_0} \alpha^t_1 \prod_{j_1=1}^{m_1} \alpha^r_{1j_1} \alpha^t_2 ... \alpha^t_n \prod_{j_n=1}^{m_n} \alpha^r_{nj_n} \tag{73}$$

Let us call $\alpha_r$ the mean cosine of the angle between two infinitesimal runs, which is much larger than $\alpha_t$ the mean cosine of the angle between two mesoscopic runs (those regarded in the previous paragraph). Then the equivalent of equation 71 is :

$$\langle \vec{a}(0) \cdot \vec{a}(t) \rangle_{\perp,\theta} = \alpha_t^n \alpha_r^m \quad \text{with} \quad m = \left\langle \sum_{k=1}^{n} m_k \right\rangle \tag{74}$$

The two Poisson processes governing respectively turns and tumbles are adiabaticly separated because of $T_r \ll T_t$ and thus independent in a mathematical sense : $P_{n,m}(t) = \left(\frac{t}{T_t}\right)^n \frac{e^{-\frac{t}{T_t}}}{n!} \left(\frac{t}{T_r}\right)^m \frac{e^{-\frac{t}{T_r}}}{m!}$.

We finally average over the number of turns and tumbles which occur between 0 and t, like in equation 72 :

$$C(t) = \sum_{n,m=0}^{\infty} \alpha_t^n \alpha_r^m P_{n,m}(t) = e^{-\frac{t}{T_t}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha t}{T_t}\right)^n e^{-\frac{t}{T_r}} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\alpha t}{T_r}\right)^m \equiv e^{-\frac{t}{\tau_c}} \tag{75}$$

In this case, the correlation time is defined by :

$$\frac{t}{\tau_c} \triangleq \frac{t}{\tau_t} + \frac{t}{\tau_r} \quad \text{where} \quad \tau_t = \frac{T_t}{1-\alpha_t} \quad \text{and} \quad \tau_r = \frac{T_r}{1-\alpha_r} \tag{76}$$

This additivity of the rates of collision looks like Matthiessen law related to metals' resistivity.
In the aftermath, we will use the simplest model with solely mesoscopic runs since we have just shown that we only have to change the time correlation to take into account rotational diffusion.

### 2.2.4 Mean square displacement

We are going to use the analogy of the polymer chain with free rotation around bounds[11]. In this sense, the mean square displacement is equivalent to the mean square end to end distance of a polymer chain.

$$\vec{r} = \sum_{i=1}^{n} \vec{l_i} \quad \Rightarrow \quad r^2 = \sum_{i=1}^{n} l_i^2 + 2 \sum_{1 \leqslant i < j \leqslant n} \vec{l_i} \cdot \vec{l_j} \tag{77}$$

[5] So as to make this paragraph easier to read, I refer to turns and tumbles with different colors.

We remind that $\alpha$ is the mean cosine of the angles between successive bonds and that the mean projection in transverse direction is zero, because the assumption of free rotation around the bonds. This yields :

$$\left\langle \vec{l}_i \cdot \vec{l}_{i+k} \right\rangle = \langle l \rangle^2 \alpha^k \tag{78}$$

We infer the mean square displacement as following :

$$\left\langle r^2 \right\rangle = n \left\langle l^2 \right\rangle + 2 \langle l \rangle^2 \sum_{1 \leqslant i < j \leqslant n} \alpha^{j-i} \tag{79}$$

In order to compute the sum, it is easy to see that j-i can take all values between 0 and n-1, and that each value k appears n-k times. Hence,

$$\sum_{1 \leqslant i < j \leqslant n} \alpha^{j-i} = \sum_{k=1}^{n-1} (n-k) \alpha^k \equiv n S_0 - S_1 \tag{80}$$

where the $S_0$ is the geometric sum and $S_1$ its derivative with respect to $\ln(\alpha)$ :

$$S_0 = \sum_{k=1}^{n-1} \alpha^k = \frac{\alpha - \alpha^n}{1 - \alpha} \tag{81}$$

$$S_1 = \sum_{k=1}^{n-1} k \alpha^k = \alpha \frac{dS_0}{d\alpha} = \alpha \frac{(1 - n\alpha^{n-1})(1-\alpha) + \alpha - \alpha^n}{(1-\alpha)^2} \tag{82}$$

We eventually take the limit $n \to +\infty$ carefully :

$$\frac{\left\langle r^2 \right\rangle}{n \left\langle l^2 \right\rangle} = 1 + \frac{2 \langle l \rangle^2}{\left\langle l^2 \right\rangle} \left( \frac{\alpha - \alpha^n}{1-\alpha} - \frac{\alpha}{n} \frac{(1 - n\alpha^{n-1})(1-\alpha) + \alpha - \alpha^n}{(1-\alpha)^2} \right) \xrightarrow[n \to +\infty]{} 1 + \frac{2 \langle l \rangle^2}{\left\langle l^2 \right\rangle} \frac{\alpha}{1-\alpha} \tag{83}$$

From equation 83, it is easy to get the formula given in reference [19] :

$$\left\langle r^2 \right\rangle = n \left\langle l^2 \right\rangle \frac{1 + \alpha \left( \frac{2 \langle l \rangle^2}{\langle l^2 \rangle} - 1 \right)}{1 - \alpha} \tag{84}$$

### 2.2.5 Expression of the diffusion coefficient

The mean square displacement in three dimensional space of Brownian particle is related to the diffusion coefficient by[7] :

$$\left\langle r(t)^2 \right\rangle = 6Dt \tag{85}$$

Then taking $t = n\mathcal{T}$, with $\mathcal{T}$ the mean time duration of a run, we get thanks to equation 84 :

$$D = \frac{\left\langle l^2 \right\rangle}{6\mathcal{T}} \frac{1 + \alpha \left( \frac{2 \langle l \rangle^2}{\langle l^2 \rangle} - 1 \right)}{1 - \alpha} \tag{86}$$

We remind that for a given direction the time duration of a run is presumed exponentially distributed. In this case, the mean square time duration S is simply twice the square of the mean time duration : $S = 2\mathcal{T}^2$. Then if we assume the same constant speed $v_0$ which is actually the activity defined at the beginning of this section 2.2 for all runs, it ensues :

$$D = \frac{v_0^2 S}{6\mathcal{T}} \frac{1 + \alpha \left( \frac{2\mathcal{T}^2}{S} - 1 \right)}{1 - \alpha} = \frac{v_0^2 \mathcal{T}}{3(1-\alpha)} \equiv \frac{v_0^2 \tau_c}{3} \tag{87}$$

In the last equality we identified the correlation defined in section 2.2.2. Nevertheless, we would like to make appear more directly the importance of rotational diffusion. That is why we expand in the second order the mean cosine of the angle between two runs :

$$\alpha = \langle \cos\theta \rangle \simeq 1 - \frac{\left\langle \theta^2 \right\rangle}{2} \tag{88}$$

On the other hand, the mean square angle[6] is related to the rotational diffusion coefficient by[20], [9] :

$$\left\langle \theta^2 \right\rangle = 4D_r\mathcal{T} \tag{89}$$

By combining equations 87, 88 and 89, we finally deduce the formula quoted in [26] :

$$D = \frac{v_0^2}{6D_r} \equiv \frac{v_0^2\tau_R}{6} \tag{90}$$

where the reorientation time $\tau_R$ is defined as the inverse of rotational diffusion coefficient.
Hence there is an implicit coupling of translation and rotation in active matter.

## 2.3 Swim stress

In this section we compute the stress of a system of active particles and define the swim pressure which is an important tool in the study of active matter.

### 2.3.1 Tensorial virial theorem

Let us consider a system of N point particles and define the position vector of one of them by $\vec{x}_\alpha = \vec{x} + \vec{r}_\alpha$, where $\vec{x}$ is barycentre's position and $\vec{r}_\alpha$ is the position of the particle in center of mass' frame. The linear momentum is as usual $\vec{p}_\alpha = m_\alpha \dot{\vec{x}}_\alpha = m_\alpha(\dot{\vec{x}} + \vec{v}_{rel\alpha})$. If $\vec{f}_\alpha$ is the force acting on particle $\alpha$, second Newton's law is simply $\dot{\vec{p}\alpha} = \vec{f}_\alpha$. With those notations, let us derive a tensorial expression of the virial theorem :

$$\frac{d}{dt}\Big(\vec{r}_\alpha \otimes \vec{p}_\alpha\Big) = \vec{v}_{rel\alpha} \otimes \vec{p}_\alpha + \vec{r}_\alpha \otimes \vec{f}_\alpha \equiv 2\bar{\bar{T}}_\alpha + \bar{\bar{W}}_\alpha \tag{91}$$

with $\bar{\bar{W}}_\alpha \triangleq \vec{r}_\alpha \otimes \vec{f}_\alpha$ the virial tensor of the particle $\alpha$ and $\bar{\bar{T}}_\alpha \triangleq \frac{1}{2}\vec{v}_{rel\alpha} \otimes \vec{p}_\alpha$ its kinetic tensor. Then we sum over all particles and take time average :

$$\left\langle \bar{\bar{W}} \right\rangle_t = -2\left\langle \bar{\bar{T}} \right\rangle_t \tag{92}$$

$\bar{\bar{W}} = \sum_\alpha \bar{\bar{W}}_\alpha$ and $\bar{\bar{T}} = \sum_\alpha \bar{\bar{T}}_\alpha$ are respectively the virial and the kinetic tensor of the system. The summation on the left hand-side of equation 91 is actually zero in average provided that the coordinates and the velocities remain finite in order to yield an upper threshold[12].
To conclude this section, let us show the kinetic tensor has a symmetry's property :

$$\bar{\bar{T}} = \frac{1}{2}\sum_\alpha \vec{v}_{rel\alpha} \otimes \vec{p}_\alpha = \frac{1}{2}\sum_\alpha \vec{v}_{rel\alpha} \otimes m_\alpha(\dot{\vec{x}} + \vec{v}_{rel\alpha}) = \frac{1}{2}\sum_\alpha m_\alpha \vec{v}_{rel\alpha} \otimes \dot{\vec{x}} + \frac{1}{2}\sum_\alpha m_\alpha \vec{v}_{rel\alpha} \otimes \vec{v}_{rel\alpha} \tag{93}$$

But $\sum_\alpha m_\alpha \vec{v}_{rel\alpha} = \vec{0}$ by definition of the center of mass. Thus we eventually get :

$$\bar{\bar{T}} = \frac{1}{2}\sum_\alpha m_\alpha \vec{v}_{rel\alpha} \otimes \vec{v}_{rel\alpha} \tag{94}$$

### 2.3.2 Stress tensor of active particles

Let us consider a system of active particles in a solvent delimited by a volume V. We try to find an expression of the swim stress $\bar{\bar{\sigma}}^{swim}$ **exerted by active particles on surrounded medium**. That is why we use a unit vector $\vec{n}$ **pointing towards the inside of volume V**. We write the force applied on particle $\alpha$ as a sum of an internal and an external force, which implies to also decompose the virial tensor in two parts. First, let us compute the external part. Each particle is submitted to a body force $\rho\vec{b}$ ($\rho$ is the density) and a stress of surface $-\bar{\bar{\sigma}}^{swim}\vec{n}$.

$$\bar{\bar{W}}^{ext} = \sum_\alpha \vec{r}_\alpha \otimes \vec{f}_\alpha^{ext} = \int_V \vec{x} \otimes \rho\vec{b}dV - \int_{\partial V} \vec{x} \otimes \bar{\bar{\sigma}}^{swim}\vec{n}dS \tag{95}$$

[6]The references provides $\left\langle \left(\vec{u}(t) - \vec{u}(0)\right)^2 \right\rangle = 4D_r\mathcal{T}$ with $\vec{u}$ a unit vector, but according to Al Kashi theorem $\left(\vec{u}(t) - \vec{u}(0)\right)^2 = 2(1 - \cos\theta)$. Then a Taylor expansion in the second order of the cosine gives equation 89.

We change the surface integral in a volume integral thanks to Gauss-Ostrogradski theorem and then use the following identity[7] :

$$\operatorname{div}(\vec{x} \otimes \bar{\bar{\sigma}}) = \bar{\bar{\sigma}}^t + \vec{x} \otimes \operatorname{div} \bar{\bar{\sigma}} \tag{96}$$

This gives :

$$\bar{\bar{W}}^{ext} = \int_V \Big( \vec{x} \otimes \rho \vec{b} - \operatorname{div}(\vec{x} \otimes \bar{\bar{\sigma}}^{swim}) \Big) dV = \int_V \Big( - \big(\bar{\bar{\sigma}}^{swim}\big)^t + \vec{x} \otimes (\rho \vec{b} - \operatorname{div} \bar{\bar{\sigma}}^{swim}) \Big) dV \tag{97}$$

However, at low Reynolds number we have the Stokes equation :

$$\rho \vec{b} - \operatorname{div} \bar{\bar{\sigma}}^{swim} = \vec{0} \tag{98}$$

so that

$$\bar{\bar{W}}^{ext} \equiv -V \left\langle \big(\bar{\bar{\sigma}}^{swim}\big)^t \right\rangle_N \tag{99}$$

with the average over all particles of the swim stress defined by $\left\langle \bar{\bar{\sigma}}^{swim} \right\rangle_N \triangleq \frac{1}{V} \int_V \bar{\bar{\sigma}}^{swim} dV$. Let us take the time average of equation 99 :

$$\left\langle \bar{\bar{\sigma}}^{swim} \right\rangle_{N,t} = -\frac{1}{V} \left( \left\langle \bar{\bar{W}}^{ext} \right\rangle_t \right)^t \tag{100}$$

Now we want to use the tensorial virial theorem proved in the previous section in order to prove that the external virial tensor is symmetric in the exchange of the vectors of outer product and that we can skip the transposition. But our active particles are a priori not points. Hence we have to choose a time scale and a length scale such that active particles can be regarded as points. In this case we can use equation 92 :

$$\left\langle \bar{\bar{W}}^{ext} \right\rangle_t = - \left\langle \bar{\bar{W}}^{int} \right\rangle_t - 2 \left\langle \bar{\bar{T}} \right\rangle_t \tag{101}$$

It is easy to see on equation 94 that the kinetic is symmetric and we are going to show it is also the case for the internal virial tensor by assuming there only pairwise interactions between particles.

$$\bar{\bar{W}}^{int} = \sum_\alpha \vec{f}_\alpha^{int} \otimes \vec{r}_\alpha = \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} \vec{f}_{\alpha\beta} \otimes \vec{r}_\alpha = \frac{1}{2} \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} \vec{f}_{\alpha\beta} \otimes \vec{r}_\alpha + \vec{f}_{\beta\alpha} \otimes \vec{r}_\beta \tag{102}$$

Then by using the third Newton's law, $\vec{f}_{\beta\alpha} = -\vec{f}_{\alpha\beta}$, we find :

$$\bar{\bar{W}}^{int} = \frac{1}{2} \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} \vec{f}_{\alpha\beta} \otimes (\vec{r}_\alpha - \vec{r}_\beta) \tag{103}$$

It is reasonable to presume that the pairwise interacting force depends only on the distance, and if we use polar coordinates we can rewrite equation 103 as following :

$$\bar{\bar{W}}^{int} = \frac{1}{2} \sum_{\substack{\alpha,\beta \\ \alpha \neq \beta}} f_{\alpha\beta}(r) \vec{e}_r \otimes \vec{r} \tag{104}$$

In this form, it is easy to see that this tensor is symmetric.
Now if we call $\vec{F}^{swim}$ the mean external force exerted on one active particle[8], we obtain an expression for the swim tensor :

$$\left\langle \bar{\bar{\sigma}}^{swim} \right\rangle_{N,t} = -\frac{1}{V} \left\langle \bar{\bar{W}}^{ext} \right\rangle_t \equiv -\frac{N}{V} \left\langle \vec{r} \otimes \vec{F}^{swim} \right\rangle_{N,t} \tag{105}$$

This derivation was inspired by reference [1].

[7]

$$\operatorname{div}(\vec{x} \otimes \bar{\bar{\sigma}}) = \operatorname{div}(x_k \sigma_{ij} \vec{e}_k \otimes \vec{e}_i \otimes \vec{e}_j) = \frac{\partial (x_k \sigma_{ij})}{\partial x_j} \vec{e}_k \otimes \vec{e}_i = \delta_{kj} \sigma_{ij} \vec{e}_k \otimes \vec{e}_i + \frac{\partial \sigma_{ij}}{\partial x_j} x_k \vec{e}_k \otimes \vec{e}_i \equiv \bar{\bar{\sigma}}^t + \vec{x} \otimes \operatorname{div} \bar{\bar{\sigma}}$$

[8] Rigorously, this average over all particles is defined by $\left\langle \vec{r} \otimes \vec{F}^{swim} \right\rangle_N \triangleq \frac{1}{N} \sum_\alpha \vec{r}_\alpha \otimes \vec{f}_\alpha^{ext}$.

## 2.4 Extensions

### 2.4.1 Commentary on [26]

In section 24 I tried to provide a derivation for equation 1 of [26] but I didn't completely succeed. Indeed, I defined the swim force like the mean external force exerted on one active particle so it is not reduced to the self-propulsive force on the contrary to definition proposed in [26]. At this stage, I would like to emphasize on something weird in [26] : the external force written in the over-damped Langevin equation on the first page is a priori NOT the swim force, but it should be since they use the definition of the swim tensor in the aftermath. Another point to highlight in [26] is the mysterious link between swim stress and diffusion coefficient. The authors wrote :

$$\vec{0} = -\gamma \vec{v}(t) + \vec{F}(t) \tag{106}$$

$$\bar{\bar{\sigma}} = -n\gamma \int \left\langle \vec{v}(t') \otimes \vec{v}(t) \right\rangle_t dt' \tag{107}$$

First, notice there is no explicit noise in equation 106 and we do not know whether there are interactions between particles. There should be at least a hard spheres potential in order to hinter overlaps, since particles are not points ( the issue of not point particles has ever been stressed in section 2.3 ). Furthermore we have shown in section 2.2.3 that motion of an active particle is properly modeled by taking into account translational and rotational diffusion. Thus we need the couple of equations 59.
Actually equation 107 is correct provided that $\vec{F}(t)$ is the so-called swim force. And, as defined in equation 106 it is perfectly coherent with my derivation in section 24. Then the authors state that :

$$\bar{\bar{D}} = \int \left\langle \vec{v}(t') \otimes \vec{v}(t) \right\rangle_t dt' \tag{108}$$

where $\bar{\bar{D}}$ is the tensor of diffusivity. This might be correct if the time correlation of elusive force $\vec{F}(t)$ is $\left\langle \vec{F}(t) \otimes \vec{F}(t') \right\rangle_t = 2D\gamma^2 \delta(t-t')\mathbf{1}$. Hence comes back the question of what is exactly the swim force.

Another point to stress on is the fact that a hard spheres potential involves an extra pressure $P_{ex} = nk_B T\, 4g(2a)\phi$. How to take it into account ?
We can notice that this formula is similar to formulas (that I am not able to explain) proposed in [26] for the non dilute limit.

### 2.4.2 Possible assessments

We have just evoked the non dilute limit but it can have other consequences. Indeed, Einstein showed that viscosity of a suspension is actually :

$$\check{\eta} = \eta \left( 1 + \frac{5}{2}\phi \right) \tag{109}$$

This formula is for rigid particles with volume fraction $\phi$ immersed in a fluid of viscosity $\eta$. It was generalized by Batchelor for fluid particles with viscosity $\eta'$ :

$$\check{\eta} = \eta \left[ 1 + \left( \frac{\eta + \frac{5}{2}\eta'}{\eta + \eta'} \right) \phi \right] \tag{110}$$

Rigorously a swimmer can not be modeled by a rigid particle since it has to move its body. For an as interesting as funny discussion of this point, see reference [23]. Therefore we should apply Batchelor formula 110 in Stokes law.

While we discuss interaction between active particles, let us talk about Hamaker's work. At microscopic scale every particles is submitted to Van der Vaals interaction, at least to London dispersion forces. However our active particles are not microscopic but rather mesoscopic : they are composed of many molecules. Then to get interaction between two swimmers we have to sum pairwise Van der Vaals interactions between molecules composing them. Hamaker did calculations for spheres but it is theoretically possible to generalize his method to bodys of any shape. Furthermore he presumed addivity of interactions but this ansatz can relaxed : it is the Lifschitz theory especially useful for dielectrics.

In section 1.1.2 we introduce Magnus effect coming from a coupling between rotation and translation. Moreover we have shown with equation 90 that such a coupling is always present in active matter. Then one can wonder whether it is relevant to write Magnus force in the Langevin equation. My answer would be no. Indeed motion of an active particle is so complicated that Magnus effect has never been detected even regarding long times. Besides adding a term $\vec{\Omega} \times \vec{v}$ would hinder writting an overdamped equation and imply a scaring Fokker-Planck equation. It also would have to update Farage's work, what may be very tricky.

In the end we can emphasize on the fact that Stokes' law is not valid at great distance for the sphere. It is better to solve Oseen equation rather Stokes equation. Thanks to the method of matching asymptotic expansions[14], [22] one can show corrections depending on the Reynolds number $Re \triangleq \frac{RU}{\nu}$ :

$$F_z = 6\pi\eta RU\left(1 + \frac{3}{8}Re + \frac{9}{40}Re^2 \ln Re + O\big(Re^2\big)\right) \tag{111}$$

# 3 Casimir effect from non trivial fluctuation spectra

Casimir effect was predicted for the first time in 1948 by the Dutch physicist Hendrick Casimir[21]. It originally referred to an attractive force between two conductive and parallel plates without any charge :

$$F_{cas} = -\mathcal{A}\frac{\hbar c\pi^2}{240L^4} \tag{112}$$

where $\mathcal{A}$ is the surface of a plate and L the distance between them.
However this force has been extended to any geometry and one actually calls now " Casimir effect " any macroscopic force induced by fluctuations. For example, one can find in the litterature many referencestextsuperscript[18], [4], [2] to the " thermal Casimir effect ". Indeed, the presence of two plates in a medium at thermal equililibrium arises some fluctuations typically from a free and massless field theory and yields an interaction proportionnal to $k_B T$ with a universal amplitude depending only on the dimension of the space :

$$F_{thcas} \triangleq -\frac{\partial \mathcal{F}}{\partial L} = -k_B T\frac{(d-1)\mathcal{A}_{d-1}}{(4\pi)^{\frac{d}{2}}L^d}\Gamma\left(\frac{d}{2}\right)\zeta(d) \tag{113}$$

We are going to introduce a new kind of Casimir effect which might be used to study non equilibrium systems.

## 3.1 Theory

In far from equilibrium systems the non-equipartition of energy leads to what is generically called " a non trivial fluctuation spectrum ". This is the spectrum of noise due to random forces, noise implying the presence of waves (classical or quantum) with different modes k. Modes which have a wavevector in [k, k+dk] contribute to the energy density by dE(k). Then a fluctuation spectrum can be represented by the function $G(k) = \frac{dE}{dk}$ and it is qualified of nontrivial if G(k) is not constant.
Pressure (or force per unit surface) exerted on a plate by waves with modes in $[k, k + \delta k]$ and an angle of incidence in $[\theta, \theta + \delta\theta]$ is :

$$\delta^2 F = G(k)\delta k \cos^2\theta \frac{\delta\theta}{2\pi} \tag{114}$$

Indeed $\delta k \cos\theta$ is the normal component of the wavevector, $2\pi$ is the solid angle of an half sphere so that $\frac{\delta\theta}{2\pi}$ is the probability for the angle of incidence to be in $[\theta, \theta + \delta\theta]$ and we can explain the second cosine by an analogy : if v is a speed and n a density, then the number of corpuscules which reach the surface $\delta S$ during $\delta t$ is $nv\delta t \cos\theta\delta S$. The integration over all possible angle of incidence ie from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ gives :

$$\delta F = \frac{1}{4}G(k)\delta k \tag{115}$$

Now let us consider again two plates separated by a distance L. All modes are allowed outside the plates and the subsequent force is :

$$F_{out} = -\int_0^{+\infty} \frac{1}{4} G(k) dk \tag{116}$$

But inside the plates modes are quantified and we have to sum with discret steps $\Delta k = \frac{\pi}{L}$ to get the subsequent force :

$$F_{in} = \frac{1}{4} \sum_{n=0}^{+\infty} G\left(\frac{n\pi}{L}\right) \frac{\pi}{L} \tag{117}$$

We finally infer a Casimir force (per unit surface):

$$F_{cas} = F_{in} + F_{out} = \frac{1}{4}\left[\frac{\pi}{L} \sum_{n=0}^{+\infty} G\left(\frac{n\pi}{L}\right) - \int_0^{+\infty} G(k) dk\right] \tag{118}$$

We expect this force is zero at thermal equilibrium since it is a kind of signature of non-equilibrium systems. According to the equipartition theorem all modes have the same energy so that the contribution to the energy density of wavevectors in $[k, k + \delta k]$ is proportionnal to the number of modes in $[k, k + \delta k]$. This number is $\frac{\text{volume of a spherical shell}}{\text{volume of a mesh}} = \frac{4\pi k^2 dk}{(\frac{\pi}{L})^3}$.
Hence this leads to $G(k) \propto k^2$ inducing $F_{cas} \equiv 0$. We study now more precisely how this force changes with the distance between plates. If the spectrum is not monotonic then there is a maximum for $k = k_{max}$. For a distance L such that $\exists n_0 \in \mathbb{N}/L = \frac{n_0\pi}{k_{max}}$ there is term in the sum which is bigger than the others : $G(k_{max})$. In this case, the Riemann sum overestimates the integral and $F_{cas} > 0$. Hence this particular L the Casimir is unexpectedly repulsive.
For very narrow spectra we are able to do a Taylor expansion at $k = k_{max}$ :

$$G(k) \simeq G(k_{max}) + G'(k_{max})(k - k_{max}) + \frac{1}{2}G''(k_{max})(k - k_{max})^2 \equiv G_0\left[1 - \left(\frac{k - k_{max}}{\nu}\right)^2\right] \quad \text{for} \quad |k - k_{max}| < \nu \tag{119}$$

We have just identified $\nu$ the half width of the peak in $\frac{G''(k_{max})}{2G_0} \triangleq -\frac{1}{\nu^2}$ which is well negative since the second derivative is evaluated at the maximum.
At this point we presume the spectrum is sufficiently narrow so that there is only one mode n checking the condition $|k - k_{max}| < \nu$. The corresponding force is :

$$F_n \triangleq \frac{\pi G_0}{4L}\left[1 - \frac{1}{\nu^2}\left(\frac{n\pi}{L} - k_{max}\right)^2\right] - \frac{G_0}{4}\int_{k_{max}-\nu}^{k_{max}+\nu}\left[1 - \left(\frac{k - k_{max}}{\nu}\right)^2\right] dk \tag{120}$$

The value of the integral is $\frac{4\nu}{3}$. Then it is easy to see that $F_n$ is maximal for $k_{max} = \frac{n\pi}{L}$ as we predicted ; with our assumptions, when this maximum is reached n is actually equal to $n_0$.

$$F_{max} = F_{n_0} = \frac{\pi G_0}{4L} - \frac{G_0 \nu}{3} = \frac{G_0 k_{max}}{4n_0} - \frac{G_0 \nu}{3} \tag{121}$$

$F_n$ is minimal when $1 - \frac{1}{\nu^2}\left(\frac{n\pi}{L} - k_{max}\right)^2 = 0 \quad \Leftrightarrow \quad L = \frac{n\pi}{\nu + k_{max}}$ and $F_{min} = -\frac{G_0\nu}{3}$ is an attractive force.
If we see $F_n$ as a function of L, we can define the half width of its peak by the difference :

$$L_{max} - L_{min} \equiv \frac{n\pi}{k_{max}} - \frac{n\pi}{\nu + k_{max}} = \frac{n\pi\nu}{k_{max}(k_{max} + \nu)} \simeq \frac{n\pi\nu}{k_{max}^2} \tag{122}$$

Thus $F_n$ is an oscillating force sometimes repulsive sometimes attractive, a behaviour completely different of others Casimir forces.

## 3.2 Applications

This theory is easy to apply : whenever you have a non equilibrium system with a non trivial fluctuation spectrum, no matter where it comes from, you can find the subsequent Casimir thanks to equation **??**. For instance we can use it to explain a strange phenomenon know since Antiquity : when two ships are closed and parallel, a mysterious force sometimes appears an make them collide. This is called Maritime Casimir effect and it is simply due to fluctuations of gravity waves because of wind between ships. The spectrum has been determined experimentally :

$$S(\omega) = \frac{ag^2}{\omega^5} e^{-b\left(\frac{\omega_0}{\omega}\right)^4} \tag{123}$$

a and b are fitted parameters ; g is the gravity constant. Using the dispersion relation $\omega = \sqrt{gk}$, this yields an attractive force fucntion of the distance L between ships as expected.

The mean subject of this internship was the application of Casimir effect to study active matter. We obtain the spectrum by a Fourier transformation of time correlation function 60 according Wiener-Khintchine theorem (dispersion relation is linear) and find for large L :

$$F_{cas} \propto -\frac{1}{6L} \pi v_0^2 \tau_r \tag{124}$$

We remind that is actually the pressure exerted on the plates. In section 2.4.1 we criticized reference [26] where is defined a swim pressure $P_{swim} \triangleq -\frac{1}{3}\mathrm{Tr}\bar{\bar{\sigma}}_{swim}$. Then combining equations 105, 90 and 108 we get :

$$P_{swim} = \frac{1}{6} n\gamma v_0^2 \tau_r \tag{125}$$

As we ever mentionned, derivation of this swim pressure is not perfectly clear. Nevertheless equations 124 and 125 are extremely simlilar although they come from two fundamentally different approaches. If I had more time, I would continue on this way and try to find their link.
Explaining physically where fluctuation spectrum of active matter comes from, except from the maths, is not obvious. As we have ever said a swimmer is not rigid because it has to move its body to swim. Then friction coefficients are actually not constant so that the spectrum could be much more complicated.

# Appendix : Pressure of interacting particles[3]

Let us consider a gas of N interacting particles in a volume V and write the virial theorem[9] : $\langle T \rangle = -\frac{1}{2}\langle W \rangle$ with

$$W = \sum_{i=1}^{N} \vec{r}_i \cdot \vec{f}_i^{ext} + \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} (\vec{r}_i - \vec{r}_j)\cdot \vec{f}_{ij} \tag{126}$$

The bracket denotes the average over all particles or on time by ergodicity. We presume that the only external force comes from pressure which is constant at equilibrium, hence :

$$\langle W_{ext} \rangle = -\oiint_S \vec{r}\cdot Pd\vec{S} = -P \iiint P \,\mathrm{div}\, \vec{r} dV = -3PV \tag{127}$$

From the equipartition theorem we have $\langle T \rangle = \frac{3}{2}k_B T$. Furthermore $\left\langle \sum_{\substack{j=1 \\ j\neq i}}^{N} (\vec{r}_i - \vec{r}_j)\cdot \vec{f}_{ij} \right\rangle$ is independent of i

since all particles are equivalent in average. We eventually get :

$$PV = N\left[k_B T + \left\langle \frac{1}{6}\sum_{j=1}^{N} (\vec{r}_1 - \vec{r}_j)\cdot \vec{f}_{1j} \right\rangle\right] \tag{128}$$

Particle 1 involves a perturbation of its neighbourhood. In other words, density is not uniform and we write as $dn(r) = \frac{N}{V} g\big(||\vec{r} - \vec{r}_1\big) d^3 r$ with $g(r)$ the so-called pairwise distribution function or correlation function of two particles.
$\frac{g(r)d^3 r}{V}$ is the probability to find a particle in the elementary volume $d^3 r$ repered by $\vec{r}$ provided that there is another in $\vec{r}_1$. Then we take particle 1 at the origin and change the discret sum into an integral as following :

$$\left\langle \sum_{j=1}^{N} (\vec{r}_1 - \vec{r}_j)\cdot \vec{f}_{1j} \right\rangle = \frac{N}{V}\int d^3 r\, g(r)\, \vec{r}\cdot \vec{f}(r) \tag{129}$$

Since the interacting force comes from a potential $u(r_{ij})$ such that $\vec{f}_{ij} = -\frac{du}{dr_{ij}}\frac{\vec{r}_i - \vec{r}_j}{r_{ij}}$, we finally get :

$$PV = N\left[k_B T - \frac{2\pi}{3}\frac{N}{V}\int_0^{+\infty} dr\, r^3 \frac{du}{dr} g(r)\right] \tag{130}$$

Thus interaction induces a pressure in addition the pressure of perfect gas.
For hard spheres potential : $u(r) = +\infty$ if r<2a and 0 otherwise, if a is the radius. In this case the extra pressure is :

$$P_{ex} = -\frac{2\pi}{3} n^2 \int_0^{+\infty} dr\, r^3 g(r)\big(-k_B T \delta(r - 2a)\big) = \frac{2\pi}{3} n^2 k_B T\, 8a^3 g(2a) \tag{131}$$

Introducing the volume fraction $\phi$, we infer :

$$P = nk_B T[1 + 4g(2a)\phi] \tag{132}$$

[9] see section 2.3.1 for notations and reference [12] for more informations.

# Conclusion and acknowledgement

On one hand, had never heard about active matter before this internship but I absolutly do not regret my choice. Indeed it is an as large as interesting topic. My first approach was to the point of view of fluids dynamics. I studied motion of a swimmer in a fluid and the main result is that self-propulsion in time average compensates viscous dissipation, so it is impossible to swim in a inviscid fluid. Then I learnt the theory of stochastic processes and used it to take into account the influence of active particles on each others. An active particle is actually not only a Brownian particle, it is more because of self-propulsion. In particular rotational diffusion is essential to model properly active matter. I have shown there is an intrinsic coupling between rotation and translation which appears in the expression of diffusion coefficient. In spite of that I think it is not necessary to take into account Magnus effect ; it would imply too much complications for a small improvement.
On the other hand, I was really impressed by the new Casimir effect developped by John. It is a great tool to study non equilibrium systems especially because we do not care where spectrum comes from. I eventually have not brought any contribution to its application to active matter. Indeed all work had been done by Farage and al in [28] where they show that active particles have a correlated Brownian motion mainly due to rotational diffusion. John had ever computed the subsequent spectrum and Casimir force. He wanted me to try assess it but it took me so much time to understand the theory that it remained almost nothing to search. I only pointed on a few details like using Oseen flow rather Stokes flow, Batchelor formula, Hamaker method to find interaction and having a look to importance of Magnus effect. John thinks there is something more in this Casimir effect for active matter and that it could related to Lifschitz theory since there are volume-volume interactions between active particles. For my part I have the feeling friction coefficients are not constant because of the swim and then fluctuation spectrum is not so simple. We sent a message to Brady asking explanation about [26] and I hope I will have the opportunity to know the final word of the story. I would like to continue studying active matter, maybe during my Phd, and in particular understand why swim pressure proposed by Brady and pressure emerging from Casimir effect are so similar.
I also thought about eventual extensions of this Casimir effect and keep many questions in mind. What does happen when geometry changes or when distance between plates is a function of time? I would like to try to do a kind of Dynamical Casimir effect ; in electrodynamics real photons are produced, could we imagine something equivalent ? In the end, the more I study the more I have questions and after this internship I am definitely sure I want to do research in theoretical physics.

My last lines are for people in Nordita. I was extremely happy to work there and you hosted me very well. Thank you to my supervisors for guiding and learning me so much things during those fantastic three monthes. A great thank you to Hans for lending me a laptop when mines died. I don't forget my colleagues in internship, Phd and postdoc : Francesco (I begin with you as I promised), Filippo, Zeyd, Satyajit, Shrikanth, Cristobal, Woosok, Stefano, Raffaele, Ginno, Lennard, Cristopher, Anna, Sergey, Matthias, Stanislav, Nishant and Akshay. I hope I will meet each of you again to discuss physics or simply to have fun, and I am looking forwards to coming back to Nordita for the Phd.
I finally would like to thank Gatien Verley (LPT) for adding my name in the publication which I worked on when I was when I did my internship with him, although I did not write it, and for supporting my application in Nordita.

## References


[1] N. C. Admal and E. B. Tadmor. A unified interpretation of stress in molecular systems. *Journal of elasticity*, **100**(63-143), 2010.

[2] D. S. Dean an A. Gopinathan. Out-of-equilibrium behavior of Casimir-type fluctuation-induced forces for free classical fields. *Physical Review E*, **81**(4), 2010.

[3] D. Lederer B. Diu, C. Guthmann and B. Roulet. *Eléments de physique statistique*. Enseignement des sciences, 1996.

[4] D. Bartolo". Fluctuations of Fluctuation-Induced Casimir-Like Forces. *Physical Review Letters*, **89**(23), 2002.

[5] H. C. Berg. *Random Walks in Biology*. Princeton University, Princeton, 1993.

[6] H. C. Berg and D. A. Brown. Chemotaxis in *Escherichia coli* analysed by Tree-simensional Tracking. *Nature, Lond.*, **239**(500), 1972.

[7] S. Chandrasekhar. Stochastic Problems in Physics and Astronomy. *Rev. Mod. Phys.*, **15**(1), 1943.

[8] S. Childress. *Mechanics of Swimming and Flying*. Cambridge Studies in Mathematical Biology. Cambridge University Press, 1981.

[9] J. K. G. Dhont. *An Introduction to Dynamics of Colloids*. Elsevier Science, Amsterdam, 1996.

[10] J. Einarsson. *Orientational dynamics of small non-spherical particles in fluid flows*. PhD thesis, University of Gothenburg. Faculty of Science, 2013.

[11] P. J. Flory. *Statistical mechanics of chain molecules*. Interscience, New-York, 1969.

[12] C. P. Poole H. Goldstein and J. L. Safko. *Classical Mechanics*. Addison Wesley, MA, 2000.

[13] A. Bonfils H. Vroylandt and G. Verley. Efficiency fluctuations of small machines with unknown losses. *Physical Review E*, **93**(5), 2016.

[14] J. Happel and H. Brenner. *Low Reynolds number hydrodynamics : with special applications to particulate media*. Mechanics of Fluids and Transport Processes. Springer Netherlands, 2012.

[15] S. Hess. Coupled Translational and Rotational Motions of a Sphere in a Fluid. *Zeitschrift für Naturforschung A*, **23**(Issue 8):1095–1101, 1968.

[16] J. Hinch, L. Petit, J.P. Hulin, and É. Guyon. *Hydrodynamique physique 3e édition (2012)*. Savoirs Actuels. EDP Sciences, 2012.

[17] H. Lamb. *Hydrodynamics*. University Press, 1916.

[18] H. Li and M. Kardar. Fluctuation-induced forces between manifolds immersed in correlated fluids. *Physical Review A*, **46**(10):6490–6500, 1992.

[19] P. S. Lovely and F. W. Dahlquist. Statistical Measures of Bacterial Motility and Chemotaxis. *J. Theor. Biol.*, **50**:477–496, 1975.

[20] M.Doi and S. F. Edwards. *The Theory of Polymer Dynamics*. Clarendon Press, Oxford, 1988.

[21] K.A. Milton. *The Casimir Effect : Physical Manifestations of Zero-point Energy*. EBL-Schweitzer. World Scientific, 2001.

[22] I. Proudman and J. R. A. Pearson. Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *Journal of Fluid Mechanics*, **2**:237–262, 5 1957.

[23] E. M Purcell. Life at low Reynolds number. *Am. J. Phys.*, **45**(1):3–11, 1977.

[24] P. Romanczuk, M. Bär, W. Ebeling, B. Lindner, and L. Schimansky-Geier. Active brownian particles. *The European Physical Journal Special Topics*, 202(**1**):1–162, 2012.

[25] S. I. Rubinow and Joseph B. Keller. The transverse force on a spinning sphere moving in a viscous fluid. *Journal of Fluid Mechanics*, **11**:447–459, 1961.

[26] W. Yan S. C. Takatori and J. F. Brady. Swim Pressure : Stress Generation in Active Matter. *Phys. Rev. Lett.*, **113**(028103), 2014.

[27] A. Solon. *Physique statistique de la matière active.* PhD thesis, Université Paris Diderot (Paris 7) - Sorbonne Paris Cité École Doctorale “Physique en Île-de-France”, 2015.

[28] P. Krinninger T. F. F. Farage and J. M. Brader. Effective interactions in active Brownian suspensions. *Phys. Rev. Lett. E*, **91**(042310), 2015.